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Hermite Interpolatory Operator Framework for Stability-Constrained Transitions Between Integer-Order Dynamical Systems

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ABSTRACT

This paper develops the theoretical framework of the Hermite interpolatory derivative operator for constructing local paths between classical integer-order dynamical models. The main idea of this work is to interpolate not only the values of the operators at integer-order nodes but also the direction of change with respect to the tuning parameter. This additional Hermite condition allows the transition between models to be controlled while keeping the operator local, finite-dimensional, and based only on ordinary derivatives. We prove exact recovery at the prescribed nodes, derive the polynomial Laplace symbol of the operator, and show that the resulting linear models have rational transfer functions. We also obtain a pole-velocity formula that links the chosen transition operators with the first-order motion of simple poles. This gives the framework a useful stability interpretation. Stability windows are then described using classical Routh-Hurwitz conditions, with special attention to possible order reduction when a leading coefficient vanishes. The proposed construction is not a fractional derivative and does not aim to reproduce memory effects. Instead, it provides a structured way to build rational, stability-aware transitions between integer-order models, electrical circuits, mechanical damping models, or model reduction where one wants rational transfer functions and local integer-order operators. A concrete transition matrix is computed with an identified stability window and representative pole trajectories as an illustrative example to show how the theoretical construction can be used in practice.

1. Introduction

Integer-order differential equations are among the most common tools for modelling dynamical systems. They appear in mechanics, control theory, electrical circuits, diffusion processes, and many other areas of applied mathematics. Their main advantage is that they are local, finite-dimensional, and compatible with classical stability and control methods. In particular, linear integer-order models usually lead to rational transfer functions, which are central in frequency-domain analysis, state-space theory, and controller design [1, 2].

In many applications, it is useful to move continuously between different integer-order models. For example, a model may pass from a static algebraic response to a first-order dissipative response, or from a first-order model to a second-order inertial model. Classical integer-order calculus does not provide a direct tuning parameter for such transitions. Fractional calculus gives one possible way to describe intermediate behaviour by allowing non-integer orders of differentiation. Fractional derivatives are widely used in models with memory, hereditary effects, anomalous diffusion, viscoelastic behaviour, and flexible control response [3].

The present paper follows a different direction. We do not construct a fractional derivative, and we do not aim to reproduce nonlocal memory. Instead, we construct local operator paths between ordinary integer-order differential operators. The proposed operators are finite linear combinations of classical derivatives. Therefore, they preserve locality, remain finite-dimensional, have polynomial Laplace symbols, and lead to rational transfer functions. This makes the framework suitable for problems where one wants a tunable model while keeping the advantages of ordinary differential equations and rational system representations.

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The basic idea comes from interpolation theory. If a finite stencil of integer orders is chosen, for example

$$S = \{0,1,2\},$$

then one can construct a parameter-dependent operator that recovers the prescribed integer-order operators at the stencil nodes. Classical interpolation and approximation methods, including barycentric Lagrange interpolation and rational interpolation, provide important tools for constructing stable and accurate approximants [4, 5]. These methods motivate the use of parameter-dependent coefficient functions in a finite-dimensional operator space.

Value interpolation alone, however, does not control how the operator path leaves each stencil node. This is a limitation when the transition itself has dynamical meaning. Hermite-type interpolation is more suitable in this setting because it allows both values and derivative data to be prescribed. Recent studies on Hermite interpolation and Hermite–Fejér interpolation show that such ideas remain active in approximation theory and numerical analysis [6-8]. Motivated by this principle, we introduce a Hermite interpolatory derivative operator that prescribes both the integer-order values and the first-order parameter directions. Value-only interpolation can recover prescribed integer-order operators, but it does not prescribe the direction in which the operator path leaves those models. Ordinary rational fitting can match a denominator at one parameter value, but it usually does not explain how this denominator belongs to a controlled path of local differential operators. Fractional-order models provide a different type of interpolation, but their meaning is tied to nonlocal memory. Hence, there is a need for a local and rational framework in which the transition direction itself is part of the model data. A concrete motivation comes from stability-aware tuning in control systems, electrical circuits, and mechanical damping models. In such settings, one may want to move from a static or first-order description to a second-order description while keeping a rational transfer function and while observing how the poles move during the transition. This is close in spirit to rational approximation and model-reduction ideas used in control, where transfer functions and pole locations play a central role [9-12].

More precisely, at each stencil node j_r, we prescribe

$$H(j_r) = D^{j_r},$$

and also

$$\frac{\partial H_\alpha}{\partial \alpha}\Big|_{\alpha=j_r} = A_r.$$

The operator A_r is called the transition operator. It determines the first-order direction in which the operator path moves away from the integer-order model D^{j_r}. This is the main difference between the Hermite IDO construction and an ordinary value-based interpolatory operator path.

This transition control has a useful spectral interpretation. For a linear model whose characteristic equation is

$$Q(\alpha,\lambda) = 0,$$

a simple pole branch $\lambda(\alpha)$ satisfies

$$\lambda'(\alpha) = -\frac{\partial_\alpha Q(\alpha,\lambda)}{\partial_\lambda Q(\alpha,\lambda)}.$$

Thus, the transition operators influence the first-order motion of the poles. This connects Hermite interpolation data with stability behaviour. In particular, the construction allows one to study how poles move when the tuning parameter changes, and whether the resulting operator path remains inside a stable region.

We also investigate stability windows for these operator paths. Since the Hermite IDO has a polynomial Laplace symbol, the associated characteristic equations are ordinary polynomials. Therefore, classical polynomial stability tools, especially the Routh–Hurwitz criterion, can be applied. Special attention is given to the second-order case, because it is the simplest setting in which static, first-order, and second-order regimes can be connected. The possibility of order reduction is also considered, since the leading coefficient may vanish for some parameter values.

The main contributions of this work are as follows. First, we introduce the Hermite interpolatory derivative operator as a local finite-dimensional path between integer-order differential operators. Second, we prove exact recovery at the stencil nodes and transition matching through the prescribed operators A_r. Third, we derive the polynomial Laplace symbol and show that the resulting linear models have rational transfer functions. Fourth, we obtain a pole-velocity formula that links transition data with the first-order motion of simple poles. Finally, we describe stability windows using classical Routh–Hurwitz conditions, including the effect of possible order reduction.

2. Preliminaries

For clarity, we mention the working assumptions used in the sequel. Functions under consideration are assumed to be sufficiently smooth for all classical derivatives that appear in the chosen stencil. When norm estimates are used, the derivatives are interpreted in the Sobolev sense. If Laplace transforms and transfer functions are used, the functions and the required derivatives are assumed to be piecewise continuous on finite intervals and are of exponential order, and homogeneous initial data are imposed whenever the transfer-function representation is written.

We recall the notation and operator setting used throughout the paper. Let $I \subseteq \mathbb{R}$ be an interval and let $u: I \to \mathbb{R}$ be sufficiently smooth. For a nonnegative integer j, we denote by

$$D^j u$$

the classical derivative of order j, with the convention

$$D^0 u = u.$$

When integral operators are needed, we denote by J^j the j-fold repeated integral operator. For $j \geq 1$, Cauchy's formula gives

$$(J^j u)(t) = \frac{1}{(j-1)!}\int_a^t (t-s)^{j-1}\, u(s)\, ds,$$

where $a \in I$ is fixed. We also set

$$J^0 u = u.$$

The operators

$$D^0, D^1, D^2, \ldots$$

and

$$J^0, J^1, J^2, \ldots$$

are the integer-order building blocks used in this paper.

Throughout the algebraic construction, the phrase sufficiently smooth means that all classical derivatives appearing in the chosen stencil exist. When norm estimates are considered later, the derivatives are understood in the Sobolev sense. When Laplace transforms are used, we assume separately that the functions and the required derivatives are piecewise continuous on finite intervals and of exponential order.

Let $\alpha \in \mathbb{R}$ be a tuning parameter. Our goal is to construct a parameter-dependent operator family whose values are finite linear combinations of integer-order operators. In the derivative case, this means that, for each fixed α, the operator has the form

$$\mathcal{D}_\alpha u = \sum_{r=0}^{m} c_r(\alpha) D^{j_r} u,$$

where $j_0, j_1, \dots, j_m$ are distinct nonnegative integers and the coefficients $c_r(\alpha)$ depend on the chosen interpolation scheme.

The finite set

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\}, \qquad j_0 < j_1 < \cdots < j_m,$$

is called an interpolation stencil. The stencil determines which integer-order operators participate in the interpolation. For example,

$$\mathcal{S} = \{0,1,2\}$$

connects static, first-order, and second-order regimes.

In this paper, an admissible stencil means a finite set of distinct nonnegative integers. No fractional or negative orders are included in the stencil.

For a fixed stencil $\mathcal{S}$, we define the finite-dimensional operator space

$$V_{\mathcal{S}} = \operatorname{span}\{D^{j_0}, D^{j_1}, \dots, D^{j_m}\}.$$

Thus, the parameter changes only the coefficients of the selected integer-order operators. It does not introduce new derivative orders or nonlocal terms.

Every interpolatory derivative operator considered in this work belongs to $V_{\mathcal{S}}$ for each fixed value of α. Hence, the operator is local and finite-dimensional.

Here, local means that the value of the operator at t depends only on u and finitely many derivatives of u at the same point t. Finite-dimensional means that all operators are taken from the fixed finite space $V_{\mathcal{S}}$.

The same construction applies to repeated integral operators. If

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\},$$

then an interpolatory integral operator may be written as

$$\mathcal{J}_\alpha u = \sum_{r=0}^{m} c_r(\alpha) J^{j_r} u.$$

In this study, the derivative case is investigated in detail because of its direct connection with dynamical systems, characteristic equations, and transfer functions. The integral case follows by the same interpolation principle. A key consequence of this finite-dimensional derivative representation appears in the Laplace domain. Assume that u and the required derivatives are of exponential order and that the corresponding initial data are homogeneous. Then

$$\mathcal{L}\{T_{\alpha,\mathcal{S}} u\}(s) = P_{\alpha,\mathcal{S}}(s) U(s),$$

where

$$P_{\alpha,\mathcal{S}}(s) = \sum_{r=0}^{m} c_r(\alpha) s^{j_r}.$$

Thus, $P_{\alpha,\mathcal{S}}(s)$ is a polynomial in s. Consequently, linear models constructed from such operators lead to rational transfer functions for each fixed value of α. The stability of these rational models will be studied later using classical polynomial stability criteria.

3. Hermite Interpolatory Operator Families

In this section, we construct a parameter-dependent operator path inside the space. The main point is that the path is not determined only by its values at the stencil nodes. We also prescribe its first-order direction with respect to the parameter. This is the role of the Hermite transition data.

Let

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\}, \qquad j_0 < j_1 < \cdots < j_m,$$

be an admissible stencil of distinct nonnegative integers. For each node j_r, we prescribe two types of data:

$$\mathcal{H}(j_r) = D^{j_r}$$

and

$$\frac{\partial \mathcal{H}_\alpha}{\partial \alpha}\Big|_{\alpha=j_r} = A_r.$$

Here A_r is called the transition operator at the node j_r. It determines the first-order direction in which the operator path leaves the integer-order operator D^{j_r}.

The subscript r labels the position of the node in the stencil, not necessarily the numerical value of the node.

To keep the construction local and finite-dimensional, we assume that

$$A_r \in V_{\mathcal{S}} = \operatorname{span}\{D^{j_0}, D^{j_1}, \dots, D^{j_m}\}.$$

Hence, for each r, there exist constants a_{rq} such that

$$A_r = \sum_{q=0}^{m} a_{rq} D^{j_q}.$$

Definition 3.1. *Let H_α be a differentiable parameter-dependent operator family with respect to α. For a node $j_r \in S$, the operator*

$$A_r = \frac{\partial H_\alpha}{\partial \alpha}\Big|_{\alpha=j_r}$$

is called the transition operator at j_r. It describes the first-order direction of the operator path as the parameter leaves the node j_r.

Let $\ell_r(\alpha)$ be the standard Lagrange basis polynomial associated with the node j_r. That is,

$$\ell_r(\alpha) = \prod_{\substack{0 \le q \le m \\ q \ne r}} \frac{\alpha - j_q}{j_r - j_q}.$$

The standard Hermite basis functions are

$$h_{r,0}(\alpha) = \big(1 - 2(\alpha - j_r)\ell_r'(j_r)\big)\ell_r(\alpha)^2,$$

and

$$h_{r,1}(\alpha) = (\alpha - j_r)\ell_r(\alpha)^2.$$

They satisfy

$$h_{r,0}(j_q) = \delta_{rq}, \qquad h_{r,0}'(j_q) = 0,$$

and

$$h_{r,1}(j_q) = 0, \qquad h_{r,1}'(j_q) = \delta_{rq}.$$

Definition 3.2. *Let $\mathcal{S} = \{j_0, j_1, \dots, j_m\}$ be an admissible stencil, and let $A_0, A_1, \dots, A_m \in V_{\mathcal{S}}$ be prescribed transition operators. The Hermite interpolatory derivative operator associated with $\mathcal{S}$ and the transition data $\{A_r\}_{r=0}^{m}$ is defined by*

$$\mathcal{H}_{\alpha,\mathcal{S}} u = \sum_{r=0}^{m} h_{r,0}(\alpha) D^{j_r} u + \sum_{r=0}^{m} h_{r,1}(\alpha) A_r u.$$

When the stencil and transition data are clear, we simply write $\mathcal{H}_\alpha u$.

Proposition 3.1. *For every fixed α, the Hermite interpolatory derivative operator $\mathcal{H}_{\alpha,\mathcal{S}}$ is linear and local.*

Proof. The operator $\mathcal{H}_{\alpha,\mathcal{S}}$ is a finite linear combination of the operators D^{j_r} and A_r. Each D^{j_r} is linear and local. Since each $A_r \in V_{\mathcal{S}}$, it is also a finite linear combination of integer-order derivatives, and therefore, linear and local. Hence $\mathcal{H}_{\alpha,\mathcal{S}}$ is linear and local. □

Theorem 3.1. *The Hermite interpolatory derivative operator satisfies*

$$\mathcal{H}_{j_q,\mathcal{S}} u = D^{j_q} u,$$

and

$$\frac{\partial}{\partial \alpha} \mathcal{H}_{\alpha,\mathcal{S}} u|_{\alpha=j_q} = A_q u$$

for every node $j_q \in \mathcal{S}$.

Proof. Using the Hermite basis properties,

$$h_{r,0}(j_q) = \delta_{rq}, \qquad h_{r,1}(j_q) = 0.$$

Therefore,

$$\mathcal{H}_{j_q,\mathcal{S}} u = \sum_{r=0}^{m} h_{r,0}(j_q) D^{j_r} u + \sum_{r=0}^{m} h_{r,1}(j_q) A_r u = D^{j_q} u.$$

Similarly,

$$h_{r,0}'(j_q) = 0, \qquad h_{r,1}'(j_q) = \delta_{rq}.$$

Thus,

$$\frac{\partial}{\partial \alpha} \mathcal{H}_{\alpha,\mathcal{S}} u|_{\alpha=j_q} = \sum_{r=0}^{m} h_{r,0}'(j_q) D^{j_r} u + \sum_{r=0}^{m} h_{r,1}'(j_q) A_r u = A_q u.$$

□

This theorem indicates that the Hermite IDO recovers the integer-order operators and also prescribes the transition direction at those operators.

Theorem 3.2. *Let $S = \{j_0, j_1, \dots, j_m\}$ be an admissible stencil, and suppose that $A_r \in V_S$ for $r = 0,1,\dots,m$. Then there exists a unique operator family $H_{\alpha,S}$, polynomial in α of degree at most $2m+1$ and taking values in V_S, such that*

$$H_{j_r,S} = D^{j_r}, \qquad \frac{\partial H_{\alpha,S}}{\partial \alpha}|_{\alpha=j_r} = A_r, \qquad r = 0,1,\dots,m.$$

Proof. Since $A_r \in V_S$, each prescribed value and each prescribed transition operator belong to the finite-dimensional space V_S. We write the desired operator family in the form

$$H_{\alpha,S} = \sum_{q=0}^{m} c_q(\alpha) D^{j_q}.$$

The interpolation conditions prescribe the value and the first derivative, with respect to α, of this V_S-valued polynomial at each node j_r. By the standard Hermite interpolation theorem applied componentwise in the basis

$$\{D^{j_0}, D^{j_1}, \dots, D^{j_m}\},$$

there exists a unique V_S-valued polynomial of degree at most $2m+1$ satisfying these conditions. The explicit formula is

$$H_{\alpha,S} u = \sum_{r=0}^{m} h_{r,0}(\alpha) D^{j_r} u + \sum_{r=0}^{m} h_{r,1}(\alpha) A_r u.$$

This proves existence and uniqueness. □

Proposition 3.2. *The Hermite IDO can be written in the form*

$$\mathcal{H}_{\alpha,\mathcal{S}} u = \sum_{q=0}^{m} c_q(\alpha) D^{j_q} u,$$

where

$$c_q(\alpha) = h_{q,0}(\alpha) + \sum_{r=0}^{m} a_{rq}\, h_{r,1}(\alpha).$$

In particular, each $c_q(\alpha)$ is a polynomial of degree at most $2m+1$.

Proof. Since $A_r \in V_{\mathcal{S}}$, we write

$$A_r = \sum_{q=0}^{m} a_{rq}\, D^{j_q}.$$

Substituting this into the definition of $\mathcal{H}_{\alpha,\mathcal{S}}$ gives

$$\mathcal{H}_{\alpha,\mathcal{S}} u = \sum_{r=0}^{m} h_{r,0}(\alpha) D^{j_r} u + \sum_{r=0}^{m} h_{r,1}(\alpha) \sum_{q=0}^{m} a_{rq}\, D^{j_q} u.$$

Collecting the coefficient of each $D^{j_q} u$, we obtain

$$\mathcal{H}_{\alpha,\mathcal{S}} u = \sum_{q=0}^{m} c_q(\alpha) D^{j_q} u,$$

where

$$c_q(\alpha) = h_{q,0}(\alpha) + \sum_{r=0}^{m} a_{rq}\, h_{r,1}(\alpha).$$

Since $h_{q,0}$ and $h_{r,1}$ are polynomials of degree at most $2m+1$, each c_q has degree at most $2m+1$. □

Thus, the Hermite extension changes the coefficient functions but not the finite operator space.

Theorem 3.3. *Let*

$$j_{\max} = \max \mathcal{S}$$

and assume that

$$u \in H^{j_{\max}}(I).$$

Let $I_\alpha \subset \mathbb{R}$ be a compact parameter interval. Then

$$\| \mathcal{H}_{\alpha,\mathcal{S}} u \|_{L^2(I)} \leq C_H(I_\alpha) \max_{0 \leq q \leq m} \| D^{j_q} u \|_{L^2(I)},$$

where

$$C_H(I_\alpha) = \sup_{\alpha \in I_\alpha} \sum_{q=0}^{m} |c_q(\alpha)|.$$

Consequently,

$$\mathcal{H}_{\alpha,\mathcal{S}} : H^{j_{\max}}(I) \to L^2(I)$$

is bounded for every fixed α.

Proof. Using the finite-dimensional representation,

$$\mathcal{H}_{\alpha,\mathcal{S}} u = \sum_{q=0}^{m} c_q(\alpha) D^{j_q} u.$$

Hence,

$$\| \mathcal{H}_{\alpha,\mathcal{S}} u \|_{L^2(I)} \le \sum_{q=0}^{m} |c_q(\alpha)| \, \| D^{j_q} u \|_{L^2(I)}.$$

Therefore,

$$\| \mathcal{H}_{\alpha,\mathcal{S}} u \|_{L^2(I)} \le \left(\sum_{q=0}^{m} |c_q(\alpha)| \right) \max_{0\le q\le m} \| D^{j_q} u \|_{L^2(I)}.$$

Since each $c_q(\alpha)$ is continuous and I_α is compact,

$$C_H(I_\alpha) = \sup_{\alpha\in I_\alpha} \sum_{q=0}^{m} |c_q(\alpha)|$$

is finite. The estimate follows. □

Proposition 3.3. *Let*

$$A_r = \sum_{q=0}^{m} a_{rq} \, D^{j_q}$$

and define

$$M_A = \max_{0\le r,q\le m} |a_{rq}|.$$

On every compact parameter interval I_α,

$$C_H(I_\alpha) \le \Lambda_{H,0}(I_\alpha) + (m+1) M_A \Lambda_{H,1}(I_\alpha),$$

where

$$\Lambda_{H,0}(I_\alpha) = \sup_{\alpha\in I_\alpha} \sum_{q=0}^{m} |h_{q,0}(\alpha)|$$

and

$$\Lambda_{H,1}(I_\alpha) = \sup_{\alpha\in I_\alpha} \sum_{r=0}^{m} |h_{r,1}(\alpha)|.$$

Proof. From the coefficient representation,

$$c_q(\alpha) = h_{q,0}(\alpha) + \sum_{r=0}^{m} a_{rq} \, h_{r,1}(\alpha).$$

Therefore,

$$\sum_{q=0}^{m} |c_q(\alpha)| \le \sum_{q=0}^{m} |h_{q,0}(\alpha)| + \sum_{q=0}^{m} \sum_{r=0}^{m} |a_{rq}| \, |h_{r,1}(\alpha)|.$$

Using

$$|a_{rq}| \le M_A,$$

we obtain

$$\sum_{q=0}^{m} |c_q(\alpha)| \le \sum_{q=0}^{m} |h_{q,0}(\alpha)| + (m+1) M_A \sum_{r=0}^{m} |h_{r,1}(\alpha)|.$$

Taking the supremum over $\alpha \in I_\alpha$ gives the stated bound. □

This estimate shows that large transition coefficients may enlarge the operator bound even though the operator remains finite-dimensional. Therefore, in applications, the transition data should be selected with stability or regularization constraints. This observation motivates the constrained recovery and stability analysis developed below.

Theorem 3.4. *Let*

$$\mathcal{S} = \{0,1,2\},$$

and let

$$\alpha_* \in (0,2)\setminus\{1\}.$$

Suppose that a target coefficient vector

$$\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2) \in \mathbb{R}^3,$$

is prescribed. Define $v(\alpha_*) = \begin{pmatrix} h_{0,1}(\alpha_*) \\ h_{1,1}(\alpha_*) \\ h_{2,1}(\alpha_*) \end{pmatrix}$. *Then* $v(\alpha_*) \ne 0$.

Moreover, there exist transition coefficients

$$a_{rq}, \qquad r,q = 0,1,2,$$

such that the Hermite coefficient functions satisfy

$$c_q(\alpha_*) = \hat{c}_q, \qquad q = 0,1,2.$$

Among all transition coefficient matrices

$$A = (a_{rq})_{0\le r,q\le 2}$$

satisfying these recovery conditions, there is a unique matrix of minimum Frobenius norm. It is given columnwise by

$$a_{\cdot q}^{\min} = \frac{\hat{c}_q - h_{q,0}(\alpha_*)}{\| v(\alpha_*) \|_2^2} v(\alpha_*), \qquad q = 0,1,2.$$

Equivalently,

$$a_{rq}^{\min} = \frac{\hat{c}_q - h_{q,0}(\alpha_*)}{\sum_{k=0}^{2} h_{k,1}(\alpha_*)^2} h_{r,1}(\alpha_*), \qquad r,q = 0,1,2.$$

Proof. For the stencil

$$\mathcal{S} = \{0,1,2\},$$

the coefficient representation of the Hermite IDO is

$$c_q(\alpha) = h_{q,0}(\alpha) + \sum_{r=0}^{2} a_{rq} \, h_{r,1}(\alpha), \qquad q = 0,1,2.$$

Evaluating at $\alpha = \alpha_*$, the recovery condition

$$c_q(\alpha_*) = \hat{c}_q$$

becomes

$$\sum_{r=0}^{2} a_{rq} \, h_{r,1}(\alpha_*) = \hat{c}_q - h_{q,0}(\alpha_*).$$

Using the vector $v(\alpha_*)$, this is

$$v(\alpha_*)^T a_{\cdot q} = \hat{c}_q - h_{q,0}(\alpha_*),$$

where

$$a_{\cdot q} = (a_{0q}, a_{1q}, a_{2q})^T.$$

Since $\alpha_* \in (0,2)\setminus\{1\}$, it is not one of the stencil nodes 0,1,2. Hence at least one of the Hermite derivative basis functions $h_{r,1}(\alpha_*)$ is nonzero, and therefore

$$v(\alpha_*) \ne 0.$$

Thus, for each q, the linear equation

$$v(\alpha_*)^T a_{\cdot q} = \hat{c}_q - h_{q,0}(\alpha_*)$$

has solutions.

The set of all solutions is an affine plane in $\mathbb{R}^3$. The unique solution of minimum Euclidean norm is the orthogonal projection onto the span of $v(\alpha_*)$, namely

$$a_{\cdot q}^{\min} = \frac{\hat{c}_q - h_{q,0}(\alpha_*)}{\| v(\alpha_*) \|_2^2} v(\alpha_*).$$

Applying this independently for $q = 0,1,2$ gives a transition matrix $A^{\min}$. Since

$$\| A \|_F^2 = \sum_{q=0}^{2} \| a_{\cdot q} \|_2^2,$$

minimizing each column norm gives the unique minimum-Frobenius-norm transition matrix. □

This result gives an algebraic recovery statement. It shows that prescribed second-order coefficients can be matched at a non-node parameter value by a suitable choice of transition data. The result does not, by itself, guarantee stability of the whole parameter path. Stability must still be checked by the stability-window conditions discussed later. The minimum-Frobenius-norm choice is useful because it removes the non-uniqueness in the transition matrix and avoids unnecessarily large transition coefficients.

The minimum-Frobenius choice also has a practical interpretation. When many transition matrices satisfy the same recovery condition, the minimum-Frobenius matrix gives the smallest transition-energy choice in coefficient space. It avoids unnecessarily large transition coefficients, reduces sensitivity of the resulting coefficient path, and provides a canonical transition matrix without adding extra arbitrary choices. This is why it is a natural regularized choice when no application-specific transition direction is prescribed.

Remark 1. *For the normalized model*

$$H_\alpha y + y = u,$$

the transfer denominator is

$$c_2(\alpha)s^2 + c_1(\alpha)s + c_0(\alpha) + 1.$$

Therefore, if a desired denominator is

$$d_2 s^2 + d_1 s + d_0,$$

then the target Hermite coefficients at

$$\alpha = \alpha^*$$

are $\hat{c}_2 = d_2, \qquad \hat{c}_1 = d_1, \qquad \hat{c}_0 = d_0 - 1.$

By Theorem 3.4, these coefficients can be recovered at a non-node parameter value by a suitable transition matrix. The stability of the resulting denominator must still be verified separately.

The Hermite construction requires transition operators A_r. These operators should reflect the intended direction of transition between integer-order models. We record three basic choices.

The transition operators are not unique. The following examples give simple choices that may be used when no application-specific transition data are available.

A forward transition is given by

$$A_r = D^{j_{r+1}} - D^{j_r},$$

when j_{r+1} exists. This choice makes the operator path leave D^{j_r} in the direction of the next higher-order operator.

A backward transition is given by

$$A_r = D^{j_r} - D^{j_{r-1}},$$

when j_{r-1} exists. This choice is useful near a boundary or when the transition is biased toward the previous regime.

For an interior node, a balanced central transition is

$$A_r = \frac{D^{j_{r+1}} - D^{j_{r-1}}}{2}.$$

This resembles a central difference with respect to the operator order and reduces directional bias.

In control-oriented applications, the transition operator may also be selected to influence the first-order motion of poles. This interpretation is developed in the spectral section below. The main role of the transition operators is therefore not to change the finite operator space $V_{\mathcal{S}}$, but to change the coefficient path inside that space. This is the mechanism that later allows Hermite IDO models to encode pole-motion and stability information while remaining local and finite-dimensional.

3.1 Illustrative Numerical Example

In this section, we introduce an example for the stencil $\mathcal{S} = \{0,1,2\}$. Take

$$\alpha_* = \frac{1}{2}$$

and consider the normalized model

$$\mathcal{H}_\alpha y + y = u.$$

At α_*, choose the target denominator

$$s^2 + 2.4s + 1.6.$$

Thus,

$$\hat{c}_2 = 1, \qquad \hat{c}_1 = 2.4, \qquad \hat{c}_0 = 0.6.$$

For $\alpha_* = 1/2$, the Hermite derivative-basis vector is

$$v(\alpha_*) = \begin{pmatrix} 9/128 \\ -9/32 \\ -3/128 \end{pmatrix}, \qquad \| v(\alpha_*) \|_2^2 = \frac{693}{8192}.$$

Using the minimum-Frobenius formula in Theorem 3.4 gives the transition matrix

$$A^{\min} = \begin{pmatrix} 159/770 & 84/55 & 117/154 \\ -318/385 & -336/55 & -234/77 \\ -53/770 & -28/55 & -39/154 \end{pmatrix}.$$

The resulting Hermite coefficient functions are

$$\begin{aligned} c_0(\alpha) &= -\frac{16}{385}\alpha^5 - \frac{139}{385}\alpha^4 + \frac{49}{22}\alpha^3 - \frac{1167}{385}\alpha^2 + \frac{159}{770}\alpha + 1, \\ c_1(\alpha) &= -\frac{322}{55}\alpha^5 + \frac{1637}{55}\alpha^4 - \frac{534}{11}\alpha^3 + \frac{1326}{55}\alpha^2 + \frac{84}{55}\alpha, \\ c_2(\alpha) &= -\frac{282}{77}\alpha^5 + \frac{1352}{77}\alpha^4 - \frac{581}{22}\alpha^3 + \frac{905}{77}\alpha^2 + \frac{117}{154}\alpha. \end{aligned}$$

At $\alpha = 1/2$, we have

$$c_2(1/2) = 1, \qquad c_1(1/2) = 2.4, \qquad c_0(1/2) + 1 = 1.6,$$

so the target denominator is recovered exactly. For $\omega = 1$, the Routh–Hurwitz conditions

$$c_2(\alpha) > 0, \qquad c_1(\alpha) > 0, \qquad c_0(\alpha) + 1 > 0$$

hold on the interval $0 < \alpha < 1$. This gives a visible stability window for the Hermite path illustrating how the transition data, coefficient path, stability inequalities, and pole motion can be computed from the above formulas. Also, shows how the formulas can be used in a concrete second-order setting. For the chosen target denominator, the minimum-Frobenius transition matrix recovers the coefficients at $\alpha_* = 1/2$, and the computed second-order stability interval includes $0 < \alpha < 1$. This does not constitute a full empirical validation. It is an illustrative calculation meant to demonstrate reproducibility and to connect the theoretical results with stability-aware model tuning.

Although the examples emphasize

$$\mathcal{S} = \{0,1,2\},$$

the construction is valid for any finite admissible stencil

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\}$$

The second-order stencil is used because it is the simplest one connecting static, first-order, and second-order regimes. Higher-order stencils may be useful when more dynamical regimes need to be connected, but they also require more transition data and stronger stability checks.

4. Laplace Symbols and Pole Motion

Here, we study the Laplace-domain form of the Hermite IDO and its effect on the poles of the associated linear models. Throughout this section, we assume that the functions and their required derivatives are piecewise continuous on finite intervals and of exponential order. We also assume homogeneous initial data when applying the Laplace transform.

Let

$$\mathcal{H}_{\alpha,\mathcal{S}}u = \sum_{q=0}^{m} c_q(\alpha) D^{j_q} u$$

be the Hermite interpolatory derivative operator associated with the stencil

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\}.$$

Assume that u and all derivatives appearing in the stencil are piecewise continuous on every finite interval and of exponential order. Under homogeneous initial data, the classical Laplace-transform identity gives

$$\mathcal{L}\{D^{j_q}u\}(s) = s^{j_q} U(s).$$

Therefore,

$$\mathcal{L}\{\mathcal{H}_{\alpha,\mathcal{S}}u\}(s) = \sum_{q=0}^{m} c_q(\alpha) s^{j_q} U(s).$$

Definition 4.1. *The polynomial*

$$P^H_{\alpha,\mathcal{S}}(s) = \sum_{q=0}^{m} c_q(\alpha) s^{j_q}$$

is called the Hermite polynomial symbol associated with $H_{\alpha,\mathcal{S}}$. Under the Laplace-transform assumptions stated above,

$$\mathcal{L}\{H_{\alpha,\mathcal{S}}u\}(s) = P^H_{\alpha,\mathcal{S}}(s) U(s).$$

Proposition 4.1. *For every fixed α, the Hermite interpolatory derivative operator has a polynomial Laplace symbol. Consequently, the linear time-invariant model*

$$\mathcal{H}_{\alpha,\mathcal{S}}y(t) + ay(t) = bu(t)$$

has the rational transfer function

$$G_\alpha(s) = \frac{Y(s)}{U(s)} = \frac{b}{P^H_{\alpha,\mathcal{S}}(s) + a},$$

provided homogeneous initial conditions are assumed.

Proof. Taking the Laplace transform of

$$\mathcal{H}_{\alpha,\mathcal{S}}y(t) + ay(t) = bu(t)$$

under homogeneous initial conditions gives

$$\left(P^H_{\alpha,\mathcal{S}}(s) + a\right) Y(s) = bU(s).$$

Hence,

$$G_\alpha(s) = \frac{Y(s)}{U(s)} = \frac{b}{P^H_{\alpha,\mathcal{S}}(s) + a}.$$

Since $P^H_{\alpha,\mathcal{S}}(s)$ is a polynomial in s, the transfer function is rational. □

The polynomial symbol corresponds directly to a finite-order ordinary differential equation. Since

$$\mathcal{H}_{\alpha,\mathcal{S}} = \sum_{q=0}^{m} c_q(\alpha) D^{j_q},$$

any equation involving $\mathcal{H}_{\alpha,\mathcal{S}}$ involves only finitely many classical derivatives.

Remark 4.1. *For each fixed α, the equation*

$$H_{\alpha,\mathcal{S}}y(t) + ay(t) = f(t)$$

is equivalent to the finite-order ordinary differential equation

$$\sum_{q=0}^{m} c_q(\alpha) D^{j_q} y(t) + ay(t) = f(t).$$

Its order is at most $j_{\max} = \max\mathcal{S}$. The order is exactly $j_{\max}$ whenever the coefficient of $D^{j_{\max}}$ is nonzero. If this leading coefficient vanishes, the effective order is lower.

The possibility of order drop is important in applications and will be treated explicitly below in the stability section.

4.1 Second-Order Rational Form

In this paper, we use the stencil

$$\mathcal{S} = \{0,1,2\}.$$

For this case, the Hermite polynomial symbol has the form

$$P^H_\alpha(s) = c_0(\alpha) + c_1(\alpha)s + c_2(\alpha)s^2.$$

A simple input-output model of the form

$$\mathcal{H}_\alpha y(t) + y(t) = u(t)$$

therefore, has the transfer function

$$G^H_\alpha(s) = \frac{1}{c_2(\alpha)s^2 + c_1(\alpha)s + c_0(\alpha) + 1}.$$

Writing

$$d_2(\alpha) = c_2(\alpha), \qquad d_1(\alpha) = c_1(\alpha), \qquad d_0(\alpha) = c_0(\alpha) + 1,$$

we obtain

$$G^H_\alpha(s) = \frac{1}{d_2(\alpha)s^2 + d_1(\alpha)s + d_0(\alpha)}.$$

This form is also shared by an ordinary second-order rational fit

$$G^{R2}(s) = \frac{1}{b_2 s^2 + b_1 s + b_0}.$$

The difference is not the algebraic denominator alone. The difference is that in the Hermite IDO framework, the coefficients d_0, d_1, d_2 are interpreted through the coefficient functions $c_q(\alpha)$ of a transition-controlled operator path. Theorem 3.4 shows that, at a non-integer benchmark value α_*, prescribed second-order coefficients can be recovered from at least one choice of Hermite transition data. This observation is used later to separate two questions, how well does a second-order rational model fit the target? and can the fitted coefficients be embedded in a Hermite operator path? The same-order rational baseline answers the approximation question: how well can a second-order rational denominator fit the chosen target? The Hermite construction answers a different structural question: can the same denominator coefficients be embedded into a transition-controlled operator path? Thus, the Hermite IDO does not claim an automatic fitting advantage at a fixed parameter value. Its added value is the interpretation of the coefficients through transition data and the possibility of studying pole motion along the parameter path.

As shown in Table 1, the comparison investigates the intended scope of the present framework. Hermite IDO is not automatically a better fixed-order fit than a rational baseline. Its advantage is structural: the same denominator coefficients are embedded in a local operator path whose transition direction is prescribed. Lagrange-only interpolation is simpler when only node recovery is needed. Padé-type and rational approximations are natural when matching a series

Table 1: Comparison between Hermite IDO and nearby alternatives.

Method	Main input data	Locality	Transfer-function form	Pole-motion information
Hermite IDO	Values at integer-order nodes and transition operators	Local, finite-dimensional	Rational, with polynomial denominator	Explicit first-order pole velocity through transition data
Lagrange-only IDO	Values at integer-order nodes only	Local, finite-dimensional	Rational, with polynomial denominator	Not directly prescribed
Same-order rational baseline	Denominator coefficients or fitting data	Usually rational	Rational by construction	Available only after the fit, not encoded by transition operators
Padé/rational approximation	Series or frequency-response matching data	Depends on the model	Rational approximant	Pole information comes from the approximant, not from an operator path

or frequency response is the main goal. Hermite IDO becomes useful when the transition direction and the stability of the path are part of the modelling question.

We now relate the Hermite transition operators to the first-order movement of characteristic roots. Consider the homogeneous linear equation

$$\mathcal{H}_{\alpha,\mathcal{S}}y(t) + ay(t) = 0.$$

Its characteristic equation is

$$P^H_{\alpha,\mathcal{S}}(\lambda) + a = 0.$$

Define

$$Q(\alpha,\lambda) = P^H_{\alpha,\mathcal{S}}(\lambda) + a.$$

The poles of the model are the roots of

$$Q(\alpha,\lambda) = 0.$$

The roots are considered in the complex plane. Thus, pole motion is studied through branches $\lambda(\alpha) \in \mathbb{C}$.

Theorem 4.1. *Let the coefficients $c_q(\alpha)$ be the Hermite coefficient polynomials constructed above, and let*

$$Q(\alpha,\lambda) = P^H_{\alpha,\mathcal{S}}(\lambda) + a.$$

Suppose that $\lambda_0 \in \mathbb{C}$ is a simple root at the node $\alpha = j_r$, that is,

$$Q(j_r,\lambda_0) = 0, \qquad \partial_\lambda Q(j_r,\lambda_0) \neq 0.$$

Then there exists a differentiable root branch $\lambda(\alpha)$, defined for α near j_r, such that $\lambda(j_r) = \lambda_0$. Moreover,

$$\lambda'(j_r) = -\frac{\partial_\alpha Q(j_r,\lambda_0)}{\partial_\lambda Q(j_r,\lambda_0)}.$$

Proof. The result follows from the implicit function theorem. Since

$$Q(j_r,\lambda_0) = 0$$

and

$$\partial_\lambda Q(j_r,\lambda_0) \neq 0,$$

there exists a differentiable root branch $\lambda(\alpha)$ near j_r such that

$$Q(\alpha,\lambda(\alpha)) = 0, \qquad \lambda(j_r) = \lambda_0.$$

Differentiating the identity

$$Q(\alpha,\lambda(\alpha)) = 0$$

with respect to α gives

$$\partial_\alpha Q(\alpha,\lambda(\alpha)) + \partial_\lambda Q(\alpha,\lambda(\alpha))\lambda'(\alpha) = 0.$$

Evaluating at $\alpha = j_r$ gives

$$\partial_\alpha Q(j_r,\lambda_0) + \partial_\lambda Q(j_r,\lambda_0)\lambda'(j_r) = 0.$$

Therefore,

$$\lambda'(j_r) = -\frac{\partial_\alpha Q(j_r,\lambda_0)}{\partial_\lambda Q(j_r,\lambda_0)}.$$

Finally, the Hermite interpolation condition gives

$$\frac{\partial \mathcal{H}_{\alpha,\mathcal{S}}}{\partial \alpha}|_{\alpha=j_r} = A_r.$$

Thus, A_r determines the first variation of the polynomial symbol at the node, and therefore determines the first-order pole velocity. □

The same argument may be viewed over $\mathbb{C}$, or equivalently over $\mathbb{R}^2$, since Q is polynomial in λ and smooth in α. The simplicity assumption on λ_0 is essential. Multiple roots require a separate perturbation analysis.

The pole-velocity formula gives the transition operators a direct spectral meaning. Near an integer node, A_r determines the first variation of the polynomial symbol and therefore the first-order displacement of each simple pole. This observation is later used to interpret stability windows along the parameter path.

5. Spectral Stability and Stability Windows

We now study how the Hermite parameter affects spectral stability and whether the roots of the characteristic polynomial remain in the open left half-plane while the parameter varies. Since the Hermite IDO has a polynomial symbol, this can be treated using classical polynomial stability criteria.

Let

$$\mathcal{H}_{\alpha,\mathcal{S}} = \sum_{q=0}^{m} c_q(\alpha) D^{j_q}$$

be a Hermite interpolatory derivative operator associated with the arbitrary stencil

$$\mathcal{S} = \{j_0, j_1, \dots, j_m\}.$$

Consider the homogeneous linear equation

$$\mathcal{H}_{\alpha,\mathcal{S}}y(t) + ay(t) = 0, \qquad a \in \mathbb{R}.$$

Under homogeneous initial conditions, the characteristic equation is

$$P^H_{\alpha,\mathcal{S}}(\lambda) + a = 0.$$

We write

$$Q_\alpha(\lambda) = P^H_{\alpha,\mathcal{S}}(\lambda) + a.$$

Thus, stability is determined by the roots of

$$Q_\alpha(\lambda) = 0.$$

Definition 5.1. *Let $I_\alpha \subseteq \mathbb{R}$ be a parameter interval. We say that I_α is a stability window for the interpolatory model if, for every $\alpha \in I_\alpha$, all roots of*

$$Q_\alpha(\lambda) = 0$$

lie in the open left half-plane

$$\{\lambda \in \mathbb{C}: \mathrm{Re}\lambda < 0\}.$$

When the leading coefficient vanishes at some value of α, the stability test is applied to the polynomial after removing the zero leading terms. Thus, stability is always understood with respect to the effective degree of Q_α.

This definition studies stability along a parameter interval but not only at the integer-order nodes.

Before applying stability criteria, one must account for possible order reduction. Although the stencil may contain a highest derivative order, the coefficient of the corresponding power of λ may vanish for some parameter values. If

$$Q_\alpha(\lambda) = \sum_{r=0}^{N} a_r\,(\alpha)\lambda^r,$$

then the effective degree is

$$d_{\mathrm{eff}}(\alpha) = \max\{r: a_r(\alpha) \neq 0\}.$$

The stability criterion must be applied to this effective polynomial. In particular, a nominally second-order model may become first order if its leading coefficient vanishes.

The effective degree at the parameter value α is

$$d_{\mathrm{eff}}(\alpha) = \max\{r: a_r(\alpha) \neq 0\}.$$

Stability must therefore be tested using the polynomial of degree $d_{\mathrm{eff}}(\alpha)$.

Remark 5.1. *If $d_{\mathrm{eff}}(\alpha) = 0$, then the characteristic equation has no dynamical root, and the model is algebraic rather than dynamic. If $d_{\mathrm{eff}}(\alpha) = 1$, the stability condition is determined by the single root of a first-order polynomial.*

This issue is relevant for interpolatory models because a coefficient may vanish at an integer node or at an intermediate parameter value.

We first focus on the lower-order cases, since they occur naturally when a leading coefficient vanishes.

If $d_{\mathrm{eff}}(\alpha) = 1$, then

$$Q_\alpha(\lambda) = a_1(\alpha)\lambda + a_0(\alpha), \qquad a_1(\alpha) \neq 0.$$

The unique root is

$$\lambda(\alpha) = -\frac{a_0(\alpha)}{a_1(\alpha)}.$$

Hence the first-order model is stable if and only if

$$\frac{a_0(\alpha)}{a_1(\alpha)} > 0.$$

For the second-order polynomial

$$Q_\alpha(\lambda) = a_2(\alpha)\lambda^2 + a_1(\alpha)\lambda + a_0(\alpha), \qquad a_2(\alpha) > 0,$$

the classical Routh-Hurwitz criterion gives stability if and only if

$$a_1(\alpha) > 0, \qquad a_0(\alpha) > 0.$$

Therefore, an interval I_α is a second-order stability window when

$$a_2(\alpha) > 0, \qquad a_1(\alpha) > 0, \qquad a_0(\alpha) > 0$$

for all $\alpha \in I_\alpha$.

Suppose that the Hermite IDO has the form

$$\mathcal{H}_\alpha y = c_0(\alpha)y + c_1(\alpha)y' + c_2(\alpha)y''.$$

For the stencil $S = \{0,1,2\}$, the general stability conditions become especially transparent.

Consider

$$\mathcal{H}_\alpha y + \omega^2 y = 0, \qquad \omega > 0.$$

The characteristic polynomial is

$$Q_\alpha(\lambda) = c_2(\alpha)\lambda^2 + c_1(\alpha)\lambda + (c_0(\alpha) + \omega^2).$$

If

$$c_2(\alpha) > 0,$$

then the model is asymptotically stable if and only if

$$c_1(\alpha) > 0, \qquad c_0(\alpha) + \omega^2 > 0.$$

Hence the second-order stability window is

$$\mathcal{W} = \{\alpha: c_2(\alpha) > 0,\ c_1(\alpha) > 0,\ c_0(\alpha) + \omega^2 > 0\}.$$

If $c_2(\alpha) = 0$ but $c_1(\alpha) \neq 0$, the effective model is first order and stability is equivalent to

$$\frac{c_0(\alpha) + \omega^2}{c_1(\alpha)} > 0.$$

Thus, the stability test must follow the effective degree of the characteristic polynomial.

5.1 Second-Order Stability Constraints

For the Hermite second-order rational form

$$G_\alpha^H(s) = \frac{1}{d_2(\alpha)s^2 + d_1(\alpha)s + d_0(\alpha)},$$

the second-order Routh-Hurwitz condition is

$$d_2(\alpha) > 0, \qquad d_1(\alpha) > 0, \qquad d_0(\alpha) > 0.$$

For a same-order rational baseline

$$G_{R2}(s) = \frac{1}{b_2 s^2 + b_1 s + b_0},$$

the same stability condition is imposed:

$$b_2 > 0, \qquad b_1 > 0, \qquad b_0 > 0.$$

Thus, any comparison between a Hermite surrogate and an ordinary rational baseline should use the same positivity constraints. In numerical work, one may impose a positive margin such as

$$d_2, d_1, d_0 \geq 10^{-4}, \qquad b_2, b_1, b_0 \geq 10^{-4},$$

to avoid marginally stable or degenerate denominators.

5.2. Local Stability and Transition Operators

The pole-velocity theorem describes the first-order motion of roots near an integer-order model. The following result records the corresponding local stability property.

Theorem 5.1. (Local Stability Preservation). *Let $j_r \in S$, and suppose that the coefficients of Q_α depend continuously on α. Assume that all roots of*

$$Q_{j_r}(\lambda) = 0,$$

lie in the open left half-plane. If the effective degree of Q_α remains constant for α near j_r, then there exists $\varepsilon > 0$ such that all roots of

$$Q_\alpha(\lambda) = 0$$

also lie in the open left half-plane whenever

$$|\alpha - j_r| < \varepsilon.$$

Proof. Since the effective degree is fixed near j_r, the characteristic polynomial does not lose its leading term in a neighbourhood of j_r. The coefficients depend continuously on α, so the roots depend continuously on the coefficients, counted with multiplicity. At $\alpha = j_r$, all roots lie in the open left half-plane. Hence there is a positive distance between the set of roots and the imaginary axis. By continuity of the roots, this distance remains positive for all α sufficiently close to j_r. Therefore, all roots remain in the open left half-plane for $|\alpha - j_r| < \varepsilon$. □

This result is local. Stability on a larger parameter interval must be verified using explicit stability-window inequalities or direct root-location tests.

In the Hermite IDO setting, the continuity assumption is automatically satisfied because the coefficients $c_q(\alpha)$ are polynomials in α.

The transition operators affect stability through the derivatives of the coefficient functions at the stencil nodes. At $\alpha = j_r$, the Hermite condition gives

$$\frac{\partial H_{\alpha,S}}{\partial \alpha}|_{\alpha=j_r} = A_r.$$

Therefore, A_r determines $\partial_\alpha Q(j_r, \lambda)$. For a simple pole λ_0, the pole-velocity formula gives

$$\lambda'(j_r) = -\frac{\partial_\alpha Q(j_r, \lambda_0)}{\partial_\lambda Q(j_r, \lambda_0)}.$$

A stability-aware choice of A_r should prevent critical poles from moving toward the imaginary axis. For example, if λ_0 is close to the imaginary axis, one may impose the local condition

$$\mathrm{Re}\lambda'(j_r) \le 0$$

or include a penalty term when

$$\mathrm{Re}\lambda'(j_r) > 0.$$

Thus, the transition data may be selected by solving a constrained problem of the form

$$\min_{A_r \in V_S} \parallel A_r \parallel^2$$

subject to

$$\mathrm{Re}\lambda_k{}'(j_r) \le 0$$

for the critical simple poles $\lambda_k(j_r)$. This does not give a global stability guarantee, but it gives a local stability-aware rule for choosing transition directions.

As a simple illustration, consider

$$\mathcal{H}_\alpha y = c_0(\alpha)y + c_1(\alpha)y' + c_2(\alpha)y''$$

and

$$\mathcal{H}_\alpha y + \omega^2 y = 0.$$

Then

$$Q_\alpha(\lambda) = c_2(\alpha)\lambda^2 + c_1(\alpha)\lambda + c_0(\alpha) + \omega^2.$$

The signs of $c_2(\alpha)$, $c_1(\alpha)$, and $c_0(\alpha) + \omega^2$ determine the applicable stability criterion. In this way, Hermite transition data enter stability analysis through the coefficient functions and the induced characteristic polynomial. This is the main stability advantage of the Hermite construction.

6. Conclusion and Discussion

This paper developed a Hermite IDO for constructing local paths between integer-order dynamical models. The construction recovers the prescribed operators at the stencil nodes and also controls the first-order transition direction throughout prescribed operators A_r. This Hermite condition is the main difference between the proposed framework and ordinary value-based interpolation.

The resulting operators remain within a fixed finite-dimensional space of ordinary derivatives. Therefore, they are local and have polynomial Laplace symbols, which leads to rational transfer functions. The transition operators also influence the first-order motion of simple poles through the pole-velocity formula. This gives the proposed framework a useful stability interpretation.

Stability windows were described using classical Routh-Hurwitz conditions. The analysis also showed that order reduction must be handled carefully when a leading coefficient vanishes. Thus, stability cannot be inferred only from the integer-order nodes; it must be checked along the parameter path. The purpose of the operators is to provide a structured rational path between integer-order models, with transition data that can be linked to pole motion and stability. Future work should focus on higher-order stencils, regularized transition choices, stronger stability constraints, and concrete control applications.

The obtained results answer the research gap in four ways. First, the Hermite construction preserves locality because the operator remains a finite linear combination of classical integer-order derivatives. Second, preserve rationality because the Laplace symbols are polynomials. Third, add transition control through the prescribed operators A_r, which is not available in value-only interpolation. Fourth, it links this transition control to pole motion and stability windows, so that the parameter path can be checked rather than assumed to be stable.

As compared to Lagrange-only interpolation, our approach requires more data, but it gives more control over the direction of the path; compared with ordinary rational fitting, it does not merely produce denominator coefficients; it embeds them into a structured operator path, and compared with Hermite–Fejér or Birkhoff-type constructions, the present form is deliberately simple and only values and first transition directions are prescribed. Other constructions may be preferable if derivative data are incomplete, over-determined, or chosen from different nodes.

This study is mainly a theoretical study; however, the introduced numerical example is illustrative rather than a full simulation operation. Future work should investigate wider numerical work, control-design examples, parameter-varying systems, model-reduction applications, and regularized methods for choosing transition operators under global stability constraints.

Conflict of Interest

The authors declare no conflicts of interest regarding the publication of this article.

References

[1] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1980.
[2] K. Ogata, *Modern Control Engineering*, 5th ed. Upper Saddle River, NJ, USA: Prentice Hall, 2010.
[3] J. A. Conejero, J. Franceschi, and E. Picó-Marco, “Fractional vs. ordinary control systems: what does the fractional derivative provide?,” *Mathematics*, vol. 10, no. 15, Art. no. 2719, 2022.
[4] M. S. Floater and K. Hormann, “Barycentric rational interpolation with no poles and high rates of approximation,” *Numer. Math.*, vol. 107, no. 2, pp. 315–331, 2007.
[5] L. N. Trefethen, *Approximation Theory and Approximation Practice*, extended ed. Philadelphia, PA, USA: SIAM, 2019.
[6] A Séguin and D. Kressner, “Hermite interpolation with retractions on manifolds,” *BIT Numer. Math.*, vol. 64, Art. no. 15, 2024.
[7] Y. Liu and D. Guo, “Optimal Hermite–Fejér interpolation of algebraic polynomials and applications and the best one-sided approximation on

the interval $[-1,1]$," *J. Math. Anal. Appl.*, vol. 536, no. 1, Art. no. 128142, 2024.
[8] K. Manrala and S. Bahadur, "On rational Hermite–Fejér interpolation," *Filomat*, vol. 39, no. 23, pp. 8229–8244, 2025.
[9] G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximants*, 2nd ed. Cambridge, U.K.: Cambridge University Press, 1996.
[10] Y. Nakatsukasa, O. Sète, and L. N. Trefethen, "The AAA algorithm for rational approximation," *SIAM J. Sci. Comput.*, vol. 40, no. 3, pp. A1494–A1522, 2018.
[11] S. Gugercin, A. C. Antoulas, and C. Beattie, "H_2 model reduction for large-scale linear dynamical systems," *SIAM J. Matrix Anal. Appl.*, vol. 30, no. 2, pp. 609–638, 2008.
[12] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. Philadelphia, PA, USA: SIAM, 2005.