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Study of Mixed Symmetry States in Even-Even Thorium Isotopes (220-230) by IBM-2 Framework

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ABSTRACT

The study of mixed-symmetry states in thorium isotopes within the IBM-2 framework is essential for achieving a comprehensive understanding of nuclear structure, characterizing collective excitations, and validating theoretical models. In this work, the proton–neutron interacting boson model (IBM-2) was employed to calculate the low-lying energy levels of even–even thorium isotopes in the mass range 220 ≤ A ≤ 230. The computations were performed using an improved version of the neutron–proton boson code (NPBOS). The Hamiltonian was constructed with parameter values optimized to provide the best fit to experimental energy levels. The model reproduced the energy spectra of $^{(220-230)}_{90}Th$ isotopes with high accuracy, yielding root mean square error (RMSE) values of 0.014 MeV, 0.006 MeV, 0.002 MeV, 0.012 MeV, 0.503 MeV, and 0.401 MeV for each isotope, respectively, when compared with experimental data. Energy states were classified into fully symmetric (FS) and mixed-symmetry (MS) states by evaluating the F-spin of each calculated level. To investigate the role of the Majorana interaction, the parameters ξ_1^+, ξ_2^+, and ξ_3^+ in the Majorana term $M_{\pi\nu}$ were allowed to vary independently, enabling the study of their differential effects. Within the U(5) limit for $^{220}_{90}Th$, the MS states ($1_1^+, 3_1^+, 2_4^+, 2_2^+$) were found to be highly sensitive to changes in the parameter ξ_2^+ , confirming a key property of mixed-symmetry states. Furthermore, MS states with $(F = F_{Max} - 1)$ can be categorized into three groups according to their distinct dependence on the excitation energies associated with ξ_1^+, ξ_2^+, and ξ_3^+.

1. Introduction

The Interacting Boson Model (IBM) is a nuclear model that has been successfully applied to describe the collective properties of medium and heavy nuclei. It was originally formulated by Arima and Iachello [1]. The model employs bosons to simulate the motion of nucleons outside a closed shell. These bosons correspond to pairs of identical valence nucleons with angular momentum 0 (for the s-state) or 2 (for the d-state). The nearest closed shell is used to determine the number of bosons, which equals the number of particle–hole pairs. If the shell is more than half full, the nucleus is treated as a boson system [2].

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Microscopic considerations strongly suggest that neutron and proton degrees of freedom should be treated separately. As a result, a more generalized version of the model, IBM-2, was introduced. This version distinguishes between proton bosons (s_π, d_π) and neutron bosons (s_ν, d_ν), each representing pairs of valence protons or neutrons. Compared with IBM-1, which treats all bosons identically, IBM-2 provides a more detailed description of nuclei with unequal proton–neutron numbers [3].

The eigenvectors of the proton and neutron bosons may couple either symmetrically or antisymmetrically. The fully symmetric (FS) states correspond to those of IBM-1, while the antisymmetric case gives rise to the well-known mixed-symmetry (MS) states. To distinguish MS from FS states, Arima et al. introduced a new quantum number, the F-spin, into IBM-2 [1,4,5]. This quantum number is analogous to nucleon isospin and characterizes neutron–proton symmetry. Proton and neutron bosons form an F-spin doublet, and the total F-spin of a nucleus is obtained by coupling individual boson F-spin states with F = 1/2. Maximum F-spin states are fully symmetric under proton–neutron exchange, whereas mixed-symmetry states correspond to lower values of F $(F < F_{Max})$.

This research contributes to a deeper understanding of the complex dynamics within the thorium isotopic chain, shedding light on the fundamental principles that govern nuclear structure and behavior. Thorium isotopes, owing to their large mass numbers, undergo rapid changes in structure and shape. Analyzing MS states within the IBM-2 framework provides insights into this evolutionary process. These isotopes are nearly ideal systems for testing mixed-symmetry states because they exhibit properties intermediate between deformed and spherical nuclei, which favors the emergence of such states [6].

The outline of this paper is as follows: Section 2 introduces the IBM-2 Hamiltonian formulation. Section 3 reviews the theoretical background of F-spin. Section 4 compares previously reported experimental and theoretical results with the present estimates and discusses the general features of Th isotopes in the mass range A = 220–230. This section also includes six tables (Tables 1–6), which present comparisons of calculated and experimental energy levels, along with classifications of the energy states into FS and MS categories based on F-spin. In addition, the effect of the Majorana parameters on the energy levels of MS states is studied within the U(5) limit for ${}^{220}_{90}Th$. The final section provides concluding remarks.

2. the IBM-2 Hamiltonian operator:

The IBM-2 Hamiltonian operator can be written as [7] :

$$\hat{H} = \hat{H}_\rho + \hat{H}_{\pi\nu} \qquad (\rho = \pi, \nu) \qquad (1)$$

Here, the operator $\hat{H}_\rho$ includes the single ρ boson energies and the two-body interaction amongst them. The operator $\hat{H}_\rho$ can be written as:

$$\hat{H}_\rho = \varepsilon_\rho \hat{n}_{d_\rho} + \hat{V}_{\rho\rho} \qquad (2)$$

Where the operator $\hat{n}_{d_\rho}$ counts the number of the d bosons of ρ boson, ε_ρ is the binding energies of the d bosons of ρ boson, and the operator $\hat{V}_{\rho\rho}$ can be written as :

$$\begin{aligned} \hat{V}_{\rho\rho} &= \sum_{L=0,2,4} \frac{1}{2}\sqrt{2L+1}\, C_L^\rho \left[(d_\rho^\dagger d_\rho^\dagger)^{(L)}(\tilde{d}_\rho \tilde{d}_\rho)^{(L)}\right]^{(0)} \\ &+ \frac{1}{2} V_0^\rho \big[(d_\rho^\dagger d_\rho^\dagger)^{(0)}(s_\rho s_\rho)^{(0)} \\ &+ (s_\rho^\dagger s_\rho^\dagger)^{(0)}(\tilde{d}_\rho \tilde{d}_\rho)^{(0)}\big]^{(0)} \\ &+ \frac{\sqrt{5}}{\sqrt{2}} V_2^\rho \big[(d_\rho^\dagger d_\rho^\dagger)^{(2)}(\tilde{d}_\rho s_\rho)^{(2)} \\ &+ (d_\rho^\dagger s_\rho^\dagger)^{(2)}(\tilde{d}_\rho \tilde{d}_\rho)^{(2)}\big]^{(0)} \\ &+ \kappa_{\rho\rho} \hat{Q}_\rho^{(2)} . \hat{Q}_\rho^{(2)} \qquad (3) \end{aligned}$$

$why C_L^{\rho}$ parameter of interaction of the identical bosons of the d state, V_0^{ρ} and V_2^{ρ} parameters of interaction of the identical bosons and d bosons with s bosons. $\hat{Q}_{\rho}^{(2)}$ is the neutron (proton) quadrupole operator, $\kappa_{\rho\rho}$ parameter of the quadrupole interaction of the identical bosons of s and d states. The operator $\hat{Q}_{\rho}^{(2)}$ can be written as:

$$\hat{Q}_{\rho}^{(2)} = (s_{\rho}^{\dagger}\tilde{d}_{\rho} + d_{\rho}^{\dagger}s_{\rho})^{(2)} + \chi_{\rho}(d_{\rho}^{\dagger}\tilde{d}_{\rho})^{(2)} \quad (4)$$

χ_{ρ} is the quadrupole parameter of the identical bosons. The two-body $\pi - \nu$ boson interaction is given by the operator $\hat{H}_{\pi\nu}$, and it can be written as:

$$\hat{H}_{\pi\nu} = \kappa\hat{Q}_{\pi}^{(2)}.\hat{Q}_{\nu}^{(2)} + \hat{M}_{\pi\nu} \quad (5)$$

κ is the quadrupole parameter of the $\pi - \nu$ bosons. $\hat{M}_{\pi\nu}$ Majorana operator that can be written as :

$$\hat{M}_{\pi\nu} = \frac{1}{2}\,\xi_2\left(d_{\nu}^{\dagger}s_{\pi}^{\dagger} - s_{\nu}^{\dagger}d_{\pi}^{\dagger}\right)^{(2)}.\left(\tilde{d}_{\nu}s_{\pi} - s_{\nu}\tilde{d}_{\pi}\right)^{(2)} - \sum_{k=1,3} 2\xi_k\left(d_{\nu}^{\dagger}d_{\pi}^{\dagger}\right)^{(k)}.\left(\tilde{d}_{\nu}\tilde{d}_{\pi}\right)^{(k)} \quad (6)$$

$\xi_k(k = 1,3)$ parameter of the interaction of $\pi - \nu$ bosons of d states, ξ_2 parameter of $\pi - \nu$ bosons of s-d states.

3. F-spin.

The strong neutron–proton interaction in the IBM-2 model Hamiltonian leads to a high degree of neutron and proton boson mixing in the eigenstates. As a result, the eigenstates cannot be identified by explicit neutron and proton quantum numbers, since these are strongly mixed. A more suitable quantum number is the F-spin, which is analogous to isospin but not identical. The fundamental group structure of F-spin is SU (2), and its three generators can be expressed explicitly as:

$$F^{+} = d_{\pi}^{\dagger}\tilde{d}_{\nu} + s_{\pi}^{\dagger}s_{\nu} \quad (7)$$

$$F^{-} = d_{\nu}^{\dagger}\tilde{d}_{\pi} + s_{\nu}^{\dagger}s_{\pi} \quad (8)$$

$$F_z = \frac{1}{2}\left(d_{\pi}^{\dagger}\tilde{d}_{\pi} + s_{\pi}^{\dagger}s_{\pi} - d_{\nu}^{\dagger}\tilde{d}_{\nu} + s_{\nu}^{\dagger}s_{\nu}\right) \quad (9)$$

A boson is regarded as a spinor in F-spin space, with each boson carrying $F = \frac{1}{2}$. The z-component for a proton boson is $M_F = \frac{1}{2}$ and $M_F = -\frac{1}{2}$ for a neutron boson. The state becomes more symmetric as the value of F increases [6,7,8].

The F-spin quantum number ($F = F_{max}$) can be transformed into a state composed exclusively of proton bosons through successive applications of the F-spin raising operator F^{+}. The total F-spin quantum number remains ($F = F_{max}$) because the raising operator does not change it. A pairwise exchange involving only proton and neutron labels does not alter this state, which consists solely of proton bosons. Therefore, IBM-2 states with ($F = F_{max}$) are referred to as full-symmetry states FS. In contrast, all other states contain at least one pair of proton and neutron bosons with ($F < F_{max}$), and are antisymmetric under pairwise proton–neutron exchange. These are called mixed-symmetry states MS [2]. The FS states have the maximum of $F - spin$ value $F_{Max} = \frac{N_{\pi}+N_{\nu}}{2}$, while the MS states are characterized by $F = F_{max} - 1, F_{max} - 2, \ldots, F_{min}$.

The F-spin character of a state$|\psi\rangle$can be qualitatively described by introducing the following measure:

$$R = \frac{\langle\psi|F^2|\psi\rangle}{F_{Max}(F_{Max}+1)} \quad (10)$$

This ratio serves as an indicator of F-spin mixing. A state can be regarded as an FS state if (R > 80%), whereas MS components are considered dominant if (R < 80%) [4].

4. Results and discussion.

The thorium isotopes $^{(220-230)}Th$ have N_π= 4 while N_ν varies from 2 to 7. The experimental and calculated energy levels, along with the corresponding F-spin values and the calculated (R) ratio for each isotope, are summarized in Tables 1–6. The energy spectra were obtained by diagonalizing the IBM-2 Hamiltonian using the NPBOS program for numerical computations [9]. The calculated values show good agreement with the available experimental data for the energy levels of each isotope. As indicated by the RMSE values, the IBM-2 results generally reproduce the experimental spectra well, although discrepancies appear for high-spin states. This arises because the model operates within a restricted Hilbert space: the total number of bosons equals half the number of nucleons outside the closed shell. While this restriction is essential to keep the calculations tractable, it limits the model's accuracy at higher excitation energies, where additional degrees of freedom would be required.

The classification of energy states is based on the R ratio and the corresponding F-spin values. The fully symmetric FS states are found at lower excitation energies, whereas the mixed-symmetry MS states ($F = F_{max} - 1$) appear at higher energies. This behavior can be understood in terms of nucleon motion: in FS states, protons and neutrons move in phase, which maximizes the attractive interaction between them and leads to more stable, lower-energy configurations. In contrast, MS states involve out-of-phase motion between protons and neutrons, which requires additional energy to maintain; thereby raising the overall energy of the state and reducing its stability.

Furthermore, some MS states correspond to ($F = F_{Max} - 2$), such as the 2_3^+ state of $^{228}_{90}Th$ and the 2_4^+ state of $^{230}_{90}Th$.

Table 1: Comparison between Exp $[10 - 15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{220}_{90}Th$.

Isotopes	Th^{220}_{90}	$F_{Max} = 3$	$F_{Max} - 1 = 2$		
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	State
0_1^+	0	0	3	85.4	**FS**
2_1^+	0.386	0.361	3	92.3	**FS**
4_1^+	0.760	0.767	3	96.0	**FS**
6_1^+	1.166	1.183	3	97.2	**FS**
8_1^+	1.598	1.578	3	93.3	**FS**
10_1^+	2.012	2.019	3	97.2	**FS**
12_1^+	2.442	2.447	3	100	**FS**
0_2^+	----	0.854	3	95.1	**FS**
2_2^+	----	1.033	2	48.9	**MS**
2_3^+	----	1.388	3	83.8	**FS**
2_4^+	----	1.759	2	51.0	**MS**
4_2^+	----	1.256	2	47.6	**MS**
6_2^+	----	1.666	2	56.0	**MS**
3_1^+	----	3.057	2	36.9	**MS**
1_1^+	----	3.215	2	48.1	**MS**
	$RMSE = 0.014 Mev$				

The computed R value provides a good description of the corresponding MS components. Tables (1-6) show the classification of states into fully symmetric states and mixed symmetric states based on the value of the ratio R. The value of F-spin is calculated from the ratio R using equation (10).

The R ratios are less than 80% for some states of $Th^{(220-230)}_{90}$ isotopes, and all of these states are mixed symmetric states; therefore, the F-spin for these states equals ($F = F_{Max} - 1$), while the states that have R ratios greater than 80% are fully symmetric and have ($F = F_{Max}$).

Table 2: Comparison between Exp $[10 - 15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{222}_{90}Th$.

Isotopes	Th^{222}_{90} $F_{Max} = 3.5$	$F_{Max} - 1 = 2.5$			
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	state
0_1^+	0	0	3.5	89.1	**FS**
2_1^+	0.183	0.183	3.5	93.1	**FS**
4_1^+	0.440	0.450	3.5	95.5	**FS**
6_1^+	0.750	0.752	3.5	95.5	**FS**
8_1^+	1.094	1.081	3.5	93.9	**FS**
10_1^+	1.461	1.460	3.5	95.8	**FS**
12_1^+	1.851	1.855	3.5	98.5	**FS**
0_2^+	----	0.953	3.5	90.7	**FS**
2_2^+	----	1.208	2.5	70.8	**MS**
2_3^+	----	1.476	2.5	66.5	**MS**
2_4^+	----	2.031	2.5	63.7	**MS**
4_2^+	----	1.328	2.5	60.1	**MS**
6_2^+	----	1.523	2.5	60.1	**MS**
3_1^+	----	3.089	2.5	51.5	**MS**
1_1^+	----	2.571	2.5	48.5	**MS**
	$RMSE = 0.006 Mev$				

Table 3: Comparison between Exp $[10-15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{224}_{90}Th$.

Isotopes	Th^{224}_{90}	$F_{Max}=4$	$F_{Max}-1=3$		
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	State
0_1^+	0	0	4	93.0	FS
2_1^+	0.098	0.099	4	94.2	FS
4_1^+	0.284	0.286	4	94.9	FS
6_1^+	0.535	0.532	4	94.8	FS
8_1^+	0.834	0.831	4	94.6	FS
10_1^+	1.174	1.178	4	96.1	FS
12_1^+	1.550	1.547	4	98.1	FS
0_2^+	----	1.249	4	81.3	FS
2_2^+	----	1.383	3	73.5	MS
2_3^+	----	1.755	3	68.8	MS
2_4^+	----	2.442	3	61.1	MS
4_2^+	----	1.447	3	62.4	MS
6_2^+	----	1.556	3	60.4	MS
3_1^+	----	3.410	3	65.8	MS
1_1^+	----	3.006	3	51.4	MS
$RMSE = 0.002Mev$					

Table 4: Comparison between Exp $[10-15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{226}_{90}Th$.

Isotopes	Th^{226}_{90}	$F_{Max}=4.5$	$F_{Max}-1=3.5$		
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	State
0_1^+	0	0	4.5	95.6	FS
2_1^+	0.072	0.075	4.5	95.5	FS
4_1^+	0.226	0.226	4.5	95.1	FS
6_1^+	0.447	0.441	4.5	94.3	FS
8_1^+	0.722	0.710	4.5	92.9	FS
10_1^+	1.040	1.030	4.5	90.7	FS
12_1^+	1.395	1.424	4.5	92.8	FS
0_2^+	----	0.800	4.5	83.5	FS
2_2^+	----	0.999	3.5	78.0	MS
2_3^+	----	1.719	3.5	78.4	MS
2_4^+	----	1.893	3.5	56.4	MS
4_2^+	----	1.202	3.5	71.3	MS
6_2^+	----	1.402	3.5	65.7	MS
3_1^+	----	2.461	3.5	76.3	MS
1_1^+	----	4.892	3.5	61.9	MS
$RMSE = 0.012Mev$					

Table 5: Comparison between Exp $[10-15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{228}_{90}Th$

Isotopes	Th^{228}_{90}	$F_{Max}=5$	$F_{Max}-1=4$		
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	State
0_1^+	0	0	5	93.7	FS
2_1^+	0.057	0.058	5	93.0	FS
4_1^+	0.186	0.187	5	91.3	FS
6_1^+	0.378	0.378	5	88.3	FS
8_1^+	0.622	0.621	5	83.7	MS
10_1^+	0.912	0.907	4	76.8	MS
12_1^+	1.239	1.247	4	67.1	MS
0_2^+	0.832	0.845	4	71.0	MS
2_2^+	0.875	0.916	4	70.0	MS
2_3^+	0.969	1.717	3	49.6	MS
2_4^+	0.980	1.930	5	86.4	FS
4_2^+	0.968	1.075	4	68.5	MS
6_2^+	1.105	1.290	4	65.6	MS
3_1^+	1.023	2.449	4	72.3	MS
1_1^+	----	2.706	4	63.3	MS
$RMSE = 0.503Mev$					

All mixed symmetry states have more MS components than the FS components, according to the computed R ratio values.

Table 6: Comparison between Exp $[10-15]$ and calculated (IBM-2) energy levels, the computed R-ratio, and the associated F-spin numbers of $^{230}_{90}Th$.

Isotopes	Th^{230}_{90}	$F_{Max}=5.5$	$F_{Max}-1=4.5$		
State	$E_{Exp}(MeV)$	$E_{IBM-2}(MeV)$	F	$R\%$	State
0_1^+	0	0	5.5	89.6	FS
2_1^+	0.053	0.053	5.5	88.4	FS
4_1^+	0.174	0.175	5.5	86.2	FS
6_1^+	0.356	0.357	5.5	82.6	FS
8_1^+	0.594	0.595	4.5	78.3	MS
10_1^+	0.879	0.873	4.5	71.5	MS
12_1^+	1.207	1.215	4.5	65.4	MS
0_2^+	0.635	0.629	4.5	79.9	MS
2_2^+	0.678	0.713	4.5	77.7	MS
2_3^+	0.781	1.292	4.5	60.0	MS
2_4^+	1.009	1.839	3.5	53.4	MS
4_2^+	0.770	0.904	4.5	75.2	MS
6_2^+	1.039	1.148	4.5	69.0	MS
3_1^+	0.826	1.954	4.5	56.8	MS
1_1^+	----	3.019	4.5	66.2	MS
$RMSE = 0.401Mev$					

4.2. The effect of Majorana parameters (ξ_1, ξ_2 *and* ξ_3) on the energy levels within the U(5) limit.

Within the framework of the IBM-2 model, the Majorana term $\widehat{M}_{\pi\nu}$ plays a crucial role in the study of mixed-symmetry MS states of certain excited energy levels. The parameters $\xi_1, \xi_2, and\ \xi_3$ are treated as independent variables, allowing the effect of each parameter on different states to be investigated. The Majorana parameters influence various MS character states but have no impact on fully symmetric FS states.

This behavior is particularly evident in the U(5) limit of the IBM-2 model, where the excitation energies of MS states associated with different irreducible representations can be expressed analytically as functions of the Majorana parameters. The Hamiltonian operator describing the interaction of different bosons in the U(5) limit is given by the following relation [16, 17]:

$$\widehat{H} = \varepsilon\left(\hat{n}_{d_\pi} + \hat{n}_{d_\nu}\right) + \widehat{M}_{\pi,\nu} \qquad (11)$$

Thus, the eigenvalues of this operator are given by the following relation:

$$E = \varepsilon n_d + \frac{1}{2}N\xi_2 + \frac{1}{2}n_d(-\xi_2 + \xi_1) + \zeta(-\xi_1 + \xi_3) \qquad (12)$$

Where ζ is a positive coefficient that depends on the number of d-bosons, and the first term describes the excitation energies of the FS states. To study the effect of the Majorana parameters on mixed-symmetry states within the vibrational limit U(5), we have determined the mixed-symmetry states of the isotope ${}^{220}_{90}Th$ as $(2_2^+, 2_4^+ 3_1^+, 1_1^+)$. The value of the F-spin for these states is approximately (F = 2); therefore, they are classified as mixed-symmetry (MS) states with (F = F$_{max}$-1).

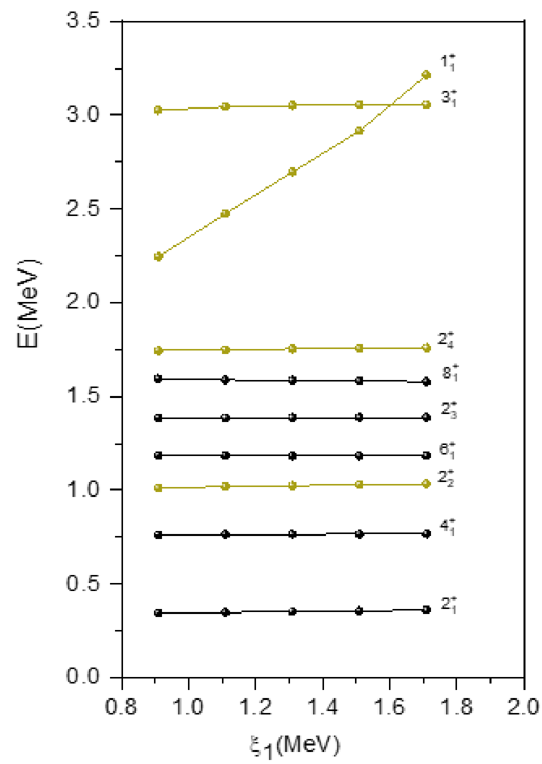

Fig.1.Excitation energies calculated for ${}^{220}_{90}Th$, as a function of ξ_1(for $\xi_2 = -0.164 MeV\ and\ \xi_3 = 1.509 MeV$).

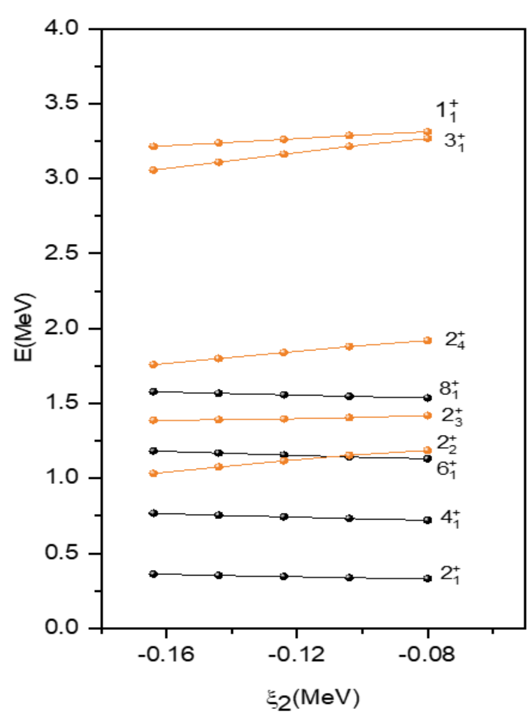

Fig.2. excitation energies calculated for ${}^{220}_{90}Th$, as a function of ξ_2 (for $\xi_1 = 1.790 MeV\ and\ \xi_3 = 1.509 MeV$).

The excitation energies of the relevant states of ${}^{220}_{90}Th$ are demonstrated by varying ξ_1 (fig. 1), ξ_2 (fig. 2), and ξ_3 (fig.3), while maintaining the other two parameters at their final adopted values. It is clear that Majorana parameters do not affect fully symmetric states ($2_1^+, 4_1^+ 6_1^+, 8_1^+, 2_3^+$), but they do affect states with mixed symmetric states ($2_2^+, 2_4^+ 3_1^+, 1_1^+$) differently.

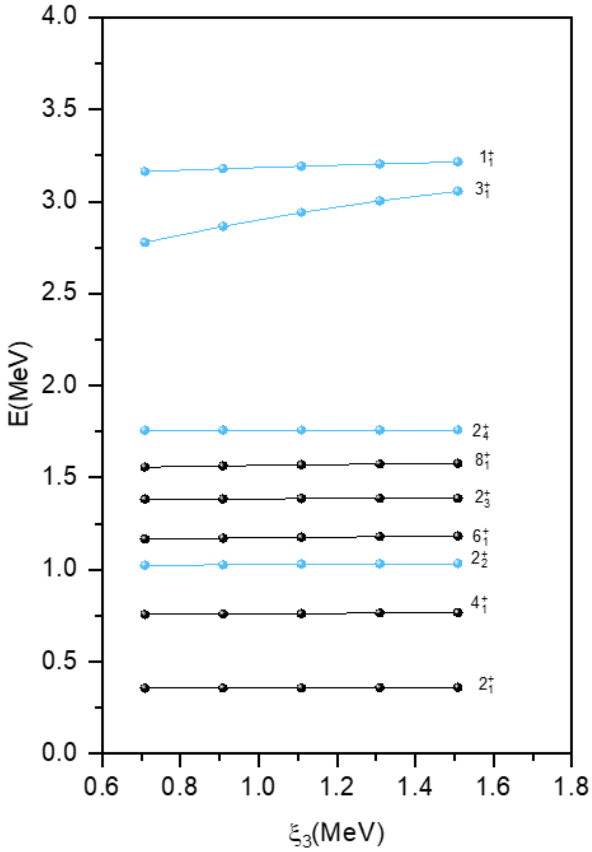

Fig.3. excitation energies calculated of ${}^{220}_{90}Th$, as a function of ξ_3 (for $\xi_2 = -0.164MeV\ and\ \xi_1 = 1.790MeV$).

The excitation energies of the MS states with ($F = F_{Max}$-1) can be classified into three groups according to the figure in reference [16]:

1. The excitation energy of MS states is:

$$E = \varepsilon n_d + \frac{1}{2} N\xi_2 \qquad (13)$$

2. The excitation energy of MS states is:

$$E = \varepsilon n_d + \frac{1}{2} N\xi_2 + \frac{1}{2} n_d(-\xi_2 + \xi_3) \qquad (14)$$

3. The excitation energy of MS states is:

$$E = \varepsilon n_d + \frac{1}{2} N\xi_2 + \frac{1}{2} n_d(-\xi_2 + \xi_1) + \zeta(-\xi_1 + \xi_3) \qquad (15)$$

As is clear from Figures 1, 2, and 3, the three groups of the MS states with the excitation energies shown in Equations (13, 14, and 15) can be classified according to their dependence on the parameters $\xi_1, \xi_2, and\ \xi_3$ into:

1. ξ_2 (all groups of mixed symmetry states).
2. ξ_3 (mixed symmetry with the excitation energy in Equations (14) and (15)).
3. ξ_1 (mixed symmetry with the excitation energy in Equation (15)) [16,17].

5. Conclusion.

This study demonstrates that the IBM2 model not only accurately describes the energy spectrum of thorium isotopes but also achieves excellent quantitative agreement, as reflected by the low RMSE values obtained. These values, ranging from 0.002 MeV to 0.503 MeV across the studied isotopes, confirm the model's validity as a robust theoretical framework for understanding the collective nuclear structure of these nuclei. The classification of the energy states of these isotopes, based on the F-spin formalism, distinguishes between fully symmetric states and mixed symmetric states. The mixed symmetric states with (F=F_max-1) can be further characterized according to their dependence on the parameters ξ_1, ξ_2, and ξ_(3). The IBM-2 model thus provides an effective predictive tool for guiding experimental research on mixed-symmetry states. The findings indicate that thorium isotopes, which exhibit transitional features between vibrational and rotational nuclear structures, offer an ideal environment for the existence of these states. Overall, this study delivers a detailed theoretical roadmap for the energies and classifications of mixed-symmetry states, thereby paving the way for future experimental verification and advancing our understanding of the collective dynamics in these nuclei.

Conflict of interest: The authors certify that there are no conflicts of interest.

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