الهيئة الليبية للبحث العلمي
Libyan Authority For Scientific Research

Libyan International Journal of Natural Sciences
LIJNS
2025
Peer-Reviewed Journal

(LIJNS) Libyan International Journal of Natural Sciences

VOLUME 1 ISSUE NO 1 MAY 2025

Bi-Annual, Peer- Reviewed, and Open Accessed e-Journal

turnitin Google Scholar

Puplished by

MILJS@aonsrt.ly

Libyan International Journal of Natural Sciences (LIJNS)
Vol.1 Issue No 1 May 2025

The Egwaider Type-III Distribution

Mohamed S. A. Muiftah Egwaider
Department of Statistics, Faculty of Science, University of Benghazi, Benghazi, Libya.

ARTICLE INFORMATION

Article history:
Received 20 April 2025
Accepted 13 May 2025
Published 15 May 2025

Keywords:
Power function distribution.
The Egwaider distribution.
Probability mass function.
Failure rate function.

A B S T R A C T

The objective of this paper was to introduce the Egwaider type-III (EGW-III) distribution, a novel semi-finite discrete probability distribution with potential real-world applications. Derived as a special case of the Egwaider distribution with semi-finite support, the properties of the suggested distribution were thoroughly investigated. Various special cases of the distribution were derived and illustrative examples featuring different parameter values were presented. Finally, the discussion and conclusion were given.

1. Introduction

The Egwaider distribution, also referred to as the discrete power function distribution, is a univariate, unimodal discrete probability distribution with three parameters. It was developed by Muiftah [Ref. 1] as a discrete counterpart to the continuous power function distribution, employing a widely recognized discretization method. This distribution is defined by its probability mass function, which takes the form:

$$P(X=x)=\begin{cases}\left(\dfrac{x-a}{s}\right)^{\theta}-\left(\dfrac{x-a+1}{s}\right)^{\theta}, & \dfrac{s}{\theta}<0\\ \left(\dfrac{x-a+1}{s}\right)^{\theta}-\left(\dfrac{x-a}{s}\right)^{\theta}, & \dfrac{s}{\theta}>0\end{cases};$$

With finite or semi-infinite support defined according to the signs of both θ and s as follows:

$$x\in[a,a+s-1],\quad s>0,\theta>0;$$
$$x\in[a+s,\ a-1],\quad s<0,\theta>0;$$
$$x\in[a+s,\infty),\quad s>0,\theta<0;$$
$$x\in(-\infty,a+s],\quad s<0,\theta<0.$$

Where a and s are integers, and θ is the shape parameter.

Thus, the Egwaider distribution can be re-written as:

$$P(Y=y)=\begin{cases}\dfrac{(y-a)^{\theta}-(y-a+1)^{\theta}}{s^{\theta}}, & \dfrac{s}{\theta}<0\\ \dfrac{(y-a+1)^{\theta}-(y-a)^{\theta}}{s^{\theta}}, & \dfrac{s}{\theta}>0\end{cases};$$

$$\begin{array}{ll} y=a,a+1,\cdots,a+s-1; & s>0,\theta>0\\ y=a+s,a+s+1,\cdots,a-1; & s<0,\theta>0\\ y=a+s,a+s+1,\cdots; & s>0,\theta<0\\ y=\cdots,a+s-1,a+s; & s<0,\theta<0\end{array}\qquad(1)$$

The first type of the Egwaider distribution (EGW-I) was obtained by Muiftah [Ref. 2], by considering $\theta>0$ and $s>0$ in the probability mass function (1), thus its probability mass function is given by:

$$P(T=t)=\frac{(t-a+1)^{\theta}-(t-a)^{\theta}}{s^{\theta}},$$
$$t=a,a+1,\cdots,a+s-1;\quad s>0;\theta>0$$

* Corresponding author: *E*-mail addresses: mohamed.muiftah@uob.edu.ly

Whereas, the second type of the Egwaider distribution (EGW-II) was obtained by Muiftah [Ref. 3], by considering $\theta > 0$ and $s < 0$ in the probability mass function (1), thus its probability mass function is given by:

$$P(W = w) = \frac{(w-a)^{\theta} - (w-a+1)^{\theta}}{s^{\theta}},$$
$$w = a+s, a+s+1, \cdots, a-1; \quad s < 0; \quad \theta > 0$$

2. Derivation of the EGW-III Distribution

Taking $\theta < 0$ and $s > 0$ in the probability mass function (1), the EGW-III distribution arises and is given by the following probability mass function:

$$P(X = x) = \frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{s^{\theta}}, \tag{2}$$
$$x = a+s, a+s+1, \cdots \quad ; \quad s > 0,\ \theta < 0$$

where, both a and s are integers.

To prove that the probability mass function (2) defined on the interval $[a+s, \infty)$, for $\theta < 0$, $s > 0$, and a, s integers, is a probability mass function, we need to verify two properties:

Property-1: Non-negativity:

We need to show that $f(x) \geq 0\ \forall$ x in $[a+s, \infty)$,

Proof:

Since $\theta < 0$, the terms $(x-a)^{\theta}$ and $(x-a+1)^{\theta}$ are positive for $x \geq a+s$ [because ($x-a \geq s > 0$) and ($x-a+1 > x-a \geq s > 0$)].

For ($\theta < 0$), the function $g(z) = z^{\theta}$ is a decreasing function. This means that if $z_1 < z_2$, then $z_1^{\theta} > z_2^{\theta}$.

Here,

$$(x-a)^{\theta} > (x-a+1)^{\theta} \quad \Rightarrow \quad (x-a)^{\theta} - (x-a+1)^{\theta} > 0,$$

thus, the numerator is positive, and since $s^{\theta} > 0$ (as $s>0$), the denominator is positive, so $f(x) \geq 0$ #.

Property-2: Normalization: We need to show that

$$\sum_{x=a+s}^{\infty} \frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{s^{\theta}} = 1,$$

Proof: Rewrite the sum:

$$\sum_{x=a+s}^{\infty} f(x) = \sum_{x=a+s}^{\infty} \frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{s^{\theta}} = \frac{1}{s^{\theta}} \sum_{x=a+s}^{\infty} [(x-a)^{\theta} - (x-a+1)^{\theta}]$$

Now, Let $k = x - a$. Since $x \in [a+s, \infty)$, $k \in [s, \infty)$, the sum becomes: $\frac{1}{s^{\theta}} \sum_{x=s}^{\infty} [k^{\theta} - (k+1)^{\theta}]$, which is a telescoping series.

$$\sum_{x=s}^{\infty} [k^{\theta} - (k+1)^{\theta}] = [s^{\theta} - (s+1)^{\theta}] + [(s+1)^{\theta} - (s+2)^{\theta}] + [(s+2)^{\theta} - (s+3)^{\theta}] + \cdots$$

Notice that most terms cancel out, leaving only the first term, and the last term which is zero [as $\lim_{x\to\infty} (k+1)^{\theta} = 0$, for $\theta < 0$].

Thus, $$\sum_{x=a+s}^{\infty} f(x) = \frac{1}{s^{\theta}} . s^{\theta} = 1 \quad \#$$

2.1. Distribution Function

The distribution function of the EGW-III distribution is:

$$F(x) = 1 - \frac{(x-a+1)^{\theta}}{s^{\theta}}, \quad x = a+s, a+s+1, \cdots; \quad s > 0,\ \theta < 0 \tag{3}$$

Proof:

$$F(x) = \sum_{v=a+s}^{x} \frac{(v-a)^{\theta} - (v-a+1)^{\theta}}{s^{\theta}}$$
$$= \frac{\{[(s)^{\theta} - (s+1)^{\theta}] + [(s+1)^{\theta} - (s+2)^{\theta}] + \cdots + [(x-a-1)^{\theta} - (x-a)^{\theta}] + [(x-a)^{\theta} - (x-a+1)^{\theta}]\}}{s^{\theta}}$$
$$= \frac{s^{\theta} - (x-a+1)^{\theta}}{s^{\theta}} = 1 - \frac{(x-a+1)^{\theta}}{s^{\theta}}. \quad \#$$

2.2. Survival Function

The survival function is the complementary function of the distribution function, hence, the survival function of the EGW-III distribution may be given as:

$$S(x) = 1 - F(x) = \frac{(x-a+1)^{\theta}}{s^{\theta}}, \tag{4}$$
$$x = a+s, a+s+1, \cdots; \quad s > 0,\ \theta < 0$$

It is clear that the EGW-III distribution reserves the same (distribution function / survival function) of the continuous power function distribution when $\theta < 0$ and s> 0 [Ref. 3,4].

2.3. Hazard Function

The hazard function of the EGW-III distribution is given by:

$$h(\mathrm{x}) = \frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{(x-a)^{\theta}}, \qquad (5)$$
$$x = a+s,\ a+s+1, \cdots; \quad s>0,\ \theta<0,$$

Proof:

$$h(\mathrm{x}) = \frac{S(x-1)-S(x)}{S(x-1)}$$
$$= \frac{\left[\frac{(x-a)^{\theta}}{s^{\theta}} - \frac{(x-a+1)^{\theta}}{s^{\theta}}\right]}{\left[\frac{(x-a)^{\theta}}{s^{\theta}}\right]}$$
$$= \frac{\left[\frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{s^{\theta}}\right]}{\left[\frac{(x-a)^{\theta}}{s^{\theta}}\right]}$$
$$= \frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{(x-a)^{\theta}} \qquad \#$$

2.4. Moments of the distribution

The moment of order r (r^{th} moment) of the EGW-III distribution is:

$$\mu'_r = E(X^r) = \frac{1}{s^{\theta}} \sum_{j=s}^{\infty} (a+j)^r \left[(j)^{\theta} - (j+1)^{\theta}\right] \qquad (6)$$

Proof:

$$\mu'_r = E(X^r) = \frac{1}{s^{\theta}} \sum_{x=a+s}^{\infty} x^r \left[(x-a)^{\theta} - (x-a+1)^{\theta}\right]$$
$$= \frac{1}{s^{\theta}} \{(a+s)^r \left[(s)^{\theta} - (s+1)^{\theta}\right]$$
$$+ (a+s+1)^r \left[(s+1)^{\theta} - (s+2)^{\theta}\right]$$
$$+ (a+s+2)^r \left[(s+2)^{\theta} - (s+3)^{\theta}\right] + \cdots\}$$
$$\therefore \quad \mu'_r = \frac{1}{s^{\theta}} \sum_{j=s}^{\infty} (a+j)^r \left[(j)^{\theta} - (j+1)^{\theta}\right] \qquad \#$$

It may observed that $\mu'_0 = E(X^0) = 1,$ as

$$\sum_{j=s}^{\infty} \left[(j)^{\theta} - (j+1)^{\theta}\right] = s^{\theta},$$

Proof: $\sum_{j=s}^{\infty} \left[(j)^{\theta} - (j+1)^{\theta}\right]$ is a telescopic summation, all terms canceled except the first term $(s)^{\theta}$ and the last term which is zero [as $\lim_{j\to\infty} (j+1)^{\theta} = 0,$ for $\theta < 0$],

$$\therefore \quad \sum_{j=s}^{\infty} \left[(j)^{\theta} - (j+1)^{\theta}\right] = \left[(s)^{\theta} - (s+1)^{\theta}\right]$$
$$+ \left[(s+1)^{\theta} - (s+2)^{\theta}\right] + \left[(s+2)^{\theta} - (s+3)^{\theta}\right] + \cdots = s^{\theta}$$

$$\Rightarrow \quad \mu'_0 = E(X^0) = 1\#$$

2.4.1. The Expected value of the distribution

The Expected value (mean) of the EGW-III distribution is the first moment of the distribution and may be calculated using the formula:

$$\mu'_1 = \mu_X = E(X) = a + \frac{\sum_{i=s}^{\infty} \left[(i)^{\theta+1} - i\,(i+1)^{\theta}\right]}{s^{\theta}}; s>0 \quad (7a)$$

Proof: Using eq. (6):

$$\mu'_1 = E(X) = \frac{1}{s^{\theta}} \sum_{j=s}^{\infty} (a+j)\left[(j)^{\theta} - (j+1)^{\theta}\right] =$$
$$\frac{1}{s^{\theta}} \sum_{i=s}^{\infty} \left[a\,(j)^{\theta} - a\,(j+1)^{\theta} + (j)^{\theta+1} - (j)(j+1)^{\theta}\right]$$
$$= \frac{1}{s^{\theta}} \{\left[a\,(s)^{\theta} - a(s+1)^{\theta} + (s)^{\theta+1} - (s)(s+1)^{\theta}\right] + \left[a\,(s+1)^{\theta}\right.$$
$$\left. - a(s+2)^{\theta} + (s+1)^{\theta+1} - (s+1)(s+2)^{\theta}\right]$$
$$+ \left[a\,(s+2)^{\theta} - a(s+3)^{\theta} + (s+2)^{\theta+1} - (s+2)(s+3)^{\theta}\right] + \cdots\}$$
$$= \frac{1}{s^{\theta}} \left[\begin{array}{l} a\,(s)^{\theta} + \{\left[(s)^{\theta+1} - (s)(s+1)^{\theta}\right] + \left[(s+1)^{\theta+1} - \right. \\ \left.(s+1)(s+2)^{\theta}\right] + \left[(s+2)^{\theta+1} - (s+2)(s+3)^{\theta}\right] + \cdots\} \end{array}\right]$$
$$= \frac{1}{s^{\theta}} \{ a\,(s)^{\theta} + \sum_{j=s}^{\infty} \left[(j)^{\theta+1} - j\,(j+1)^{\theta}\right]\}$$
$$= a + \frac{\sum_{i=s}^{\infty} \left[(j)^{\theta+1} - j\,(j+1)^{\theta}\right]}{s^{\theta}} \qquad \#$$

The sum in eq. (7a) is difficult to evaluate directly, but we can use the fact that the probability mass function is a power-law distribution. For $\theta < 0,$ the mean exists only if $\theta < -1.$ If $\theta < -1,$ the first moment (mean) is approximated as Ref. [5,6]:

$$\mu'_1 = \mu_X = E(X) \approx a + s + \frac{s}{\theta+1}; \quad s > 0. \qquad (7b)$$

Proof:

$$E(X) = \sum_{x=a+s}^{\infty} x.\frac{(x-a)^{\theta} - (x-a+1)^{\theta}}{s^{\theta}}$$
$$= \frac{1}{s^{\theta}} \sum_{x=a+s}^{\infty} x.\left[(x-a)^{\theta} - (x-a+1)^{\theta}\right]$$

Let $k = x - a$, so $x = a + k$, and the sum becomes:

$$E(X) = \frac{1}{s^{\theta}} \sum_{x=s}^{\infty} (a+k)[k^{\theta} - (k+1)^{\theta}]$$

$$= \frac{a}{s^{\theta}} \sum_{x=s}^{\infty} [k^{\theta} - (k+1)^{\theta}] + \frac{1}{s^{\theta}} \sum_{x=s}^{\infty} k[k^{\theta} - (k+1)^{\theta}]$$

The first sum is telescoping, $\sum_{x=s}^{\infty} [k^{\theta} - (k+1)^{\theta}] = s^{\theta}$,

thus: $\frac{a}{s^{\theta}} \sum_{x=s}^{\infty} [k^{\theta} - (k+1)^{\theta}] = \frac{a}{s^{\theta}} . s^{\theta} = a.$

The second sum is to be approximated using integration for large s:
For $\theta < 0$, the dominant term comes from the behavior of $k^{\theta+1}$,

$$\sum_{x=s}^{\infty} k^{\theta+1} [1 - (1+\frac{1}{k})^{\theta}] \approx \sum_{x=s}^{\infty} k^{\theta+1} (-\frac{\theta}{k}) = -\theta \sum_{x=s}^{\infty} k^{\theta}$$

$$= -\theta \int_{s}^{\infty} k^{\theta} dk = -\theta (\frac{s^{\theta+1}}{\theta+1}), \quad (\text{for} \quad \theta < -1)$$

Thus, the second sum is to be approximated as: $\frac{1}{s^{\theta}} \sum_{x=s}^{\infty} k[k^{\theta} - (k+1)^{\theta}] \approx \frac{1}{s^{\theta}} [-\theta (\frac{s^{\theta+1}}{\theta+1})] = \frac{-\theta s}{\theta+1}$
Adding the two sums, we get the approximation of the mean as:

$$E(X) \approx a + s + \frac{s}{\theta+1}; \quad s > 0, \quad \theta < -1 \quad \#$$

2.4.2. The Second Moment of the distribution

The second moment of the EGW-III distribution is given by:

$$\mu_2' = E(X^2) = a^2 + \frac{\sum_{j=s}^{\infty} \{(2a+j)[(j)^{\theta+1} - j(j+1)^{\theta}]\}}{s^{\theta}} \quad (8a)$$

Proof:

$$\mu_2' = E(X^2) = \frac{1}{s^{\theta}} \sum_{j=s}^{\infty} (a+j)^2 [(j)^{\theta} - (j+1)^{\theta}]$$

$$= \frac{1}{s^{\theta}} \sum_{i=s}^{-1} (a^2 + 2aj + j^2)[(j)^{\theta} - (j+1)^{\theta}]$$

$$= \frac{1}{s^{\theta}} \sum_{i=s}^{\infty} [a^2 (j)^{\theta} - a^2 (j+1)^{\theta} + 2a(j)^{\theta+1}$$
$$- 2aj(j+1)^{\theta} + (j)^{\theta+2} - j^2 (j+1)^{\theta}]$$

$$= \frac{1}{s^{\theta}} \{[a^2 (s)^{\theta} - a^2 (s+1)^{\theta} + 2a(s)^{\theta+1}$$
$$- 2a(s)(s+1)^{\theta} + (s)^{\theta+2} - (s)^2 (s+1)^{\theta}]$$
$$+ [a^2 (s+1)^{\theta} - a^2 (s+2)^{\theta} + 2a(s+1)^{\theta+1}$$
$$- 2a(s+1)(s+2)^{\theta} + (s+1)^{\theta+2} - (s+1)^2 (s+2)^{\theta}] + \cdots\}$$

$$= \frac{a^2 (s)^{\theta}}{s^{\theta}} +$$
$$\frac{1}{s^{\theta}} \{[2a(s)^{\theta+1} - 2a(s)(s+1)^{\theta} + (s)^{\theta+2} - (s)^2 (s+1)^{\theta}]$$
$$+ [2a(s+1)^{\theta+1} - 2a(s+1)(s+2)^{\theta} + (s+1)^{\theta+2}$$
$$- (s+1)^2 (s+2)^{\theta}] + \cdots\}$$

$$= a^2 + \frac{\sum_{j=s}^{\infty} \{2a(j)^{\theta+1} - 2aj(j+1)^{\theta} + (j)^{\theta+2} - j^2 (j+1)^{\theta}\}}{s^{\theta}}$$

$$= a^2 + \frac{\sum_{j=s}^{\infty} \{(2a+j)[(j)^{\theta+1} - j(j+1)^{\theta}]\}}{s^{\theta}} \quad \#$$

The sum in eq. (8a) is again difficult to evaluate directly, but we can use the fact that the probability mass function is a power-low distribution and integral approximation techniques. For $\theta < 0$, the second moment exists only if $\theta < -2$. If $\theta < -2$, the second moment can also be approximated using integration for large s Ref. [5,6]:

$$\mu_2' = E(X^2) \approx a^2 + \theta(\frac{s^2}{\theta+2} + \frac{2as}{\theta+1}); s > 0, \theta < -2. \quad (8b)$$

Proof: Let $k = x - a$, so $x = a + k$, the support becomes $k \in [s, \infty)$, and the probability mass function (2) in terms of k is:

$$P(K = k) = \frac{k^{\theta} - (k+1)^{\theta}}{s^{\theta}}, \quad k = s, s+1, \cdots; s > 0, \ \theta < 0$$

Since $x = a + k$, we have:

$$E(X^2) = E[(K+a)^2] = E(K^2) + 2a\, E(K) + a^2$$

We need to compute $E(K)$ and $E(K^2)$.

$$E(K) = \sum_{k=s}^{\infty} k.P(K=k) = \sum_{k=s}^{\infty} k. \frac{k^{\theta} - (k+1)^{\theta}}{s^{\theta}}$$

This is a telescopic series.
Approximating the sum for large s (using an integral approximation for $\theta < 0$):

$$E(K) = \frac{1}{s^{\theta}} \int_{s}^{\infty} k.[k^{\theta} - (k+1)^{\theta}] dk$$

For $\theta < 0$, $(k+1)^{\theta} \approx k^{\theta} + \theta k^{\theta-1}$ (first-order Taylor expansion), so: $k^{\theta} - (k+1)^{\theta} \approx -\theta k^{\theta-1}$,
thus,

$$E(K) \approx \frac{-\theta}{s^{\theta}}\int_{s}^{\infty} k.k^{\theta-1}\,dk = \frac{-\theta}{s^{\theta}}\int_{s}^{\infty} k^{\theta}\,dk$$

$$= \frac{-\theta}{s^{\theta}}\left(\frac{k^{\theta+1}}{\theta+1}\right|_{s}^{\infty} = \frac{-\theta}{s^{\theta}}\left(-\frac{s^{\theta+1}}{\theta+1}\right)$$

(since $\theta+1<0$), therefore:

$$E(K) \approx \frac{\theta s}{\theta+1}.$$

Similarly,

$$E(K^2) = \sum_{k=s}^{\infty} k^2.P(K=k) = \sum_{k=s}^{\infty} k.P(K=k)$$

$$= \sum_{k=s}^{\infty} k.\frac{k^{\theta}-(k+1)^{\theta}}{s^{\theta}},$$

Again, using the approximation:

$$k^{\theta} - (k+1)^{\theta} \approx -\theta k^{\theta-1}$$

$$E(K^2) \approx \frac{-\theta}{s^{\theta}}\int_{s}^{\infty} k^2.k^{\theta-1}\,dk = \frac{-\theta}{s^{\theta}}\int_{s}^{\infty} k^{\theta+1}\,dk$$

$$= \frac{-\theta}{s^{\theta}}\left(\frac{k^{\theta+2}}{\theta+2}\right|_{s}^{\infty} = \frac{-\theta}{s^{\theta}}\left(-\frac{s^{\theta+2}}{\theta+2}\right)$$

(since $\theta+2<0$)

Therefore:

$$E(K^2) \approx \frac{\theta s^2}{\theta+2}.$$

Thus,

$$E(X^2) = (K^2) + 2a\,E(K) + a^2$$

$$= \frac{\theta s^2}{\theta+2} + 2a\frac{\theta s}{\theta+1} + a^2 = \theta\left(\frac{s^2}{\theta+2} + \frac{2as}{\theta+1}\right) + a^2 \quad \#$$

The approximation improves as *s* becomes large.

2.5. The Variance of the distribution

Using eq. (7a) and eq. (8a) in the formula $\text{Var}(X) = E(X^2) - [E(X)]^2$, the variance of the EGW-III distribution may be given as:

$$\text{Var}(X) = \frac{\sum_{j=s}^{\infty}[(j)^{\theta+2} - j^2(j+1)^{\theta}]}{s^{\theta}} - \frac{\left[\sum_{i=s}^{\infty}[(j)^{\theta+1} - j(j+1)^{\theta})\right]^2}{s^{2\theta}} \quad (9a)$$

Proof:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\Rightarrow \text{Var}(X) = a^2 + \frac{\sum_{j=s}^{\infty}\{(2a+j)[(j)^{\theta+1} - j(j+1)^{\theta}]\}}{s^{\theta}} - \left[a + \frac{\sum_{i=s}^{\infty}[(j)^{\theta+1} - j(j+1)^{\theta}]}{s^{\theta}}\right]^2$$

Let $c_j = (j)^{\theta+1} - j(j+1)^{\theta}$, (9b)

then,

$$\text{Var}(X) = a^2 + \frac{\sum_{j=s}^{\infty}(2a+j)c_j}{s^{\theta}} - a^2 - \frac{2a\sum_{i=s}^{\infty}c_j}{s^{\theta}} - \left(\frac{\sum_{j=s}^{\infty}c_j}{s^{\theta}}\right)^2$$

$$= \frac{2a\sum_{j=s}^{-1}cj}{s^{\theta}} + \frac{\sum_{j=s}^{-1}jc_j}{s^{\theta}} - \frac{2a\sum_{j=s}^{-1}c_j}{s^{\theta}} - \frac{j\left(\sum_{j=s}^{-1}c_j\right)^2}{s^{2\theta}}$$

$$\therefore \quad \text{Var}(X) = \frac{\sum_{j=s}^{\infty}jc_j}{s^{\theta}} - \frac{\left(\sum_{j=s}^{\infty}c_j\right)^2}{s^{2\theta}} \quad (9c)$$

replacing c_j by its value given by (9b) in (9c),

$$\text{Var}(X) = \frac{\sum_{j=s}^{\infty}[(j)^{\theta+2} - j^2(j+1)^{\theta}]}{s^{\theta}} - \frac{\left[\sum_{j=s}^{\infty}[(j)^{\theta+1} - j(j+1)^{\theta})\right]^2}{s^{2\theta}} \quad \#$$

The sum in eq. (9a) is again difficult to evaluate directly, then, the variance can be approximated using the first and second moments given by (7b) and (8b), then, the variance of the EGW-III distribution will be:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\approx [\theta(\frac{s^2}{\theta+2} + \frac{2as}{\theta+1}) + a^2] - (a + s + \frac{s}{\theta+1})^2$$

$$= \frac{\theta s^2}{\theta+2} - \frac{s(4a+3s+s\theta)}{\theta+1} - \frac{s^2}{(\theta+1)^2}. \quad (9d)$$

2.6. The Mode of the distribution

The mode of the EGW-III distribution is the smallest value in the support of the distribution and is given as:

$$\text{Mode }(X) = a+s, \qquad \theta<0, s>0 \qquad (10)$$

Proof:

Since s > 0, the probability mass function (2) is a decreasing function in terms of *x*, it attains its maximum value if the numerator $[(x-a)^{\theta}-(x-a+1)^{\theta}]$ is maximum for $x \geq a+s$, and since $\theta < 0$, the negative exponent causes the terms to decrease as x grows.

To determine where the probability mass function (6) maximizes, we can examine the behavior of the differences between consecutive values of the probability mass function by determining the relative sizes of $P(X=x)$ and $P(X=x+1)$ for successive values of x:
Considering the ratio:

$$\frac{P(X=x+1)}{P(X=x)}=\frac{\dfrac{(x-a+1)^{\theta}-(x-a+2)^{\theta}}{s^{\theta}}}{\dfrac{(x-a)^{\theta}-(x-a+1)^{\theta}}{s^{\theta}}}=\frac{(x-a+1)^{\theta}-(x-a+2)^{\theta}}{(x-a)^{\theta}-(x-a+1)^{\theta}}$$

since $\theta < 0$, the function $f(x)=(x-a)^{\theta}$ is a decreasing function in x. Hence, the numerator and the denominator are both positive, therefore, $P(X=x)$ is a decreasing function of x for large x. Thus, the mode will occur at the smallest value of x in the support of the distribution which is $x = a+s$.

$$\Rightarrow Max\, P(X=x)=P(X=a+s)=\frac{(a+s-a)^{\theta}-(a+s-a+1)^{\theta}}{s^{\theta}}=\frac{(s)^{\theta}-(s+1)^{\theta}}{s^{\theta}}, \quad s>0, \theta<0.$$

It may be noticed that the mode of this distribution is analogues to that of the continuous power function distribution when θ < 0 and s > 0 and to that of the EGW(II) distribution.

2.7. The Median of the distribution

The median of a discrete distribution is the value m such that: $P(X \leq m) \geq 0.5$ and $P(X \geq m) \geq 0.5$.
The median m is the smallest integer such that $F(m) \geq 0.5$, where, $F(.)$ is the distribution function of the EGW-III distribution given by equation (3). Since the probability mass function (2) decreases as x increases, therefore, the distribution function $F(m)$ will increase as m increases, and the median will be around the point where the sum of the probabilities first reaches or exceeds 0.5, i.e. the median will be near the smaller values of x, and is expected to be close to $a+s$. However, for exact determination, we would need to compute the cumulative sum of the probabilities.

For large x, the probabilities decrease rapidly, and the median is likely to be just a few steps larger than $a+s$ because the probabilities decrease quickly.

Thus, the median will be near $a+s$, but it will depend on the specific values of θ and s. For small values of θ, we might expect the median to be $a+s+1$ or $a+s+2$.

3. Special Cases of the Egwaider Type-III Distribution

Here are some discrete distributions that may be considered special cases from the EGW-III distribution:

3.1. Taking $a=0, s=1$, the EGW-III distribution reduces to the discrete **Pareto distribution** (also known as **Lomax distribution**), with the probability mass function:

$$P(X=x)=x^{\theta}-(x+1)^{\theta}, \qquad x\in[1,\infty), \qquad (11)$$

which is a heavy-tailed distribution often used to model phenomena with power-law behavior.

3.2. Considering $a = 0, s = 1, \theta = -1$, the EGW-III distribution reduces to the **Zipf distribution** with a probability mass function:

$$P(X=x)=\frac{1}{x}-\frac{1}{x+1}, \qquad x\in[1,\infty), \qquad (12)$$

which is used to model rank-frequency data, such as word frequencies in natural language.

3.3. For $a = 0, s = 1, \theta = -k$ for some $k > 0$, the EGW-III distribution reduces to the **generalized Zipf distribution** with a probability mass function:

$$P(X=x)=x^{-k}-(x+1)^{-k}, \qquad x\in[1,\infty), \quad (13)$$

which extends the Zipf distribution to allow for different tail behaviors.

3.4. Putting $a = 0, s = 1, \theta = -\beta$ for some $\beta > 0$, in the probability mass function (2), the EGW-III distribution reduces to the **discrete Weibull distribution** with a probability mass function:

$$P(X=x)=x^{-\beta}-(x+1)^{-\beta}, x\in[1,\infty), \qquad (14)$$

which is used to model failure times in reliability analysis.

3.5. Shifted distributions

The EGW-III distribution given by the probability mass function (2) shifted by a. For example, if $a=1, s=1, \theta=-1$, the probability mass function becomes:

$$P(X=x)=\frac{1}{x}-\frac{1}{x+1}, \qquad x\in[2,\infty), \qquad (15)$$

This is a shifted version of the Zipf distribution in which $a = 0$.

4. Practical Applications

4.1. Table 1 displays the probability mass function [P(x)], distribution function [F(x)], survival function [S(x)], and hazard function [h(x)] for the EGW-III (0, 10, -0.2) distribution, whereas, Figures 1 and 2 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{x^{-0.2}-(x+1)^{-0.2}}{(10)^{-0.2}}, \quad x=10,11,\cdots$$

Table 1: of the EGW-III (0, 10, -0.2) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
10	0.0188815	0.0188815	0.9811185	0.0188815
11	0.0169259	0.0358075	0.96419250	0.0172517
12	0.0153124	0.0511199	0.94888008	0.0158810
13	0.0139602	0.0650801	0.93491988	0.0147123
⋮	⋮	⋮	⋮	⋮
100	0.0012544	0.3702970	0.62970294	0.0019881
⋮	⋮	⋮	⋮	⋮
1000	0.0000794	0.6019724	0.39802759	0.0001998
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

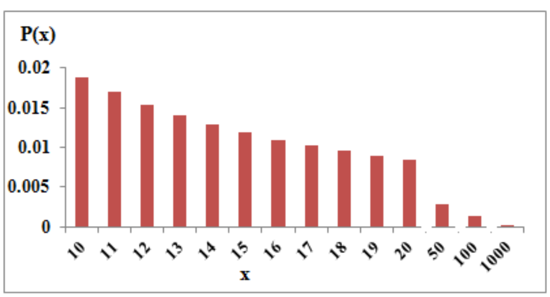

Fig. 1: The probability mass function of the EGW-III (0, 10. -0.2) distribution

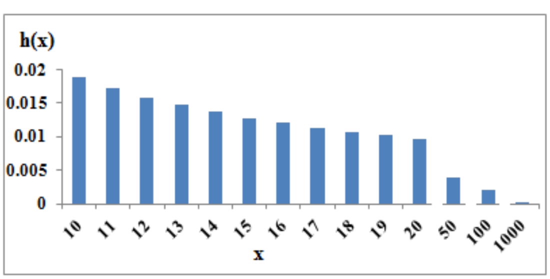

Fig. 2: The hazard function of the EGW-III (0, 10. -0.2) distribution

4.2. Table 2 displays the probability mass function [$P(x)$], the distribution function [$F(x)$], the survival function [$S(x)$], and the hazard function[$h(x)$] of the EGW-III (5, 8, -0.2) distribution, whereas, figures 3 and 4 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{(x-5)^{-0.2}-(x-4)^{-0.2}}{(8)^{-0.2}}, \quad x=13,14,\cdots$$

Table 2: The EGW-III (5, 8, -0.2) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
13	0.0232813 2	0.0232813	0.9767186	0.0232813
14	0.0203661 8	0.0436475	0.9563525	0.0208516
⋮	⋮	⋮	⋮	⋮
50	0.0031049 7	0.2951991	0.7048008	0.0043861
⋮	⋮	⋮	⋮	⋮
100	0.0012754 1	0.3916356	0.6083643	0.0020920
⋮	⋮	⋮	⋮	⋮
100 0	0.0000765	0.6189638 9	0.3810361 1	0.0002008 8
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

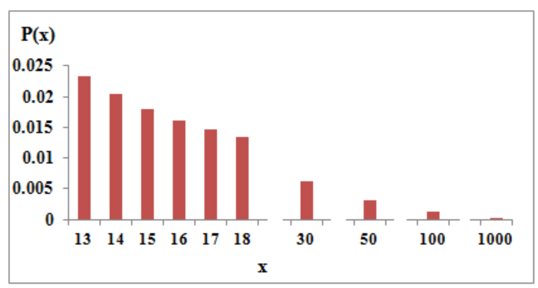

Fig. 3: The probability mass function of the EGW-III (5, 8. -0.2) distribution

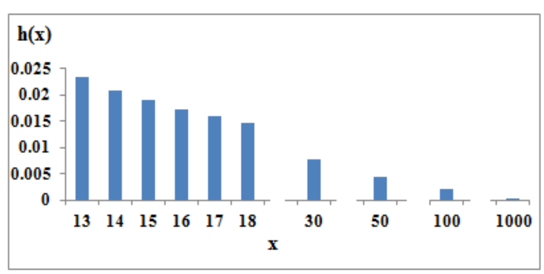

Fig. 4: The hazard function of the EGW-III (5, 8. -0.2) distribution

4.3. Table 3 displays the probability mass function [$P(x)$], the distribution function [$F(x)$], the survival function [$S(x)$], and the hazard function[$h(x)$] of the EGW-III (0, 10, -1.3) distribution, whereas, figures 5 and 6 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{x^{-1.3}-(x-1)^{-1.3}}{(10)^{-1.3}}, \qquad x=10,11,\cdots$$

Table 3: The EGW-III (0, 10, -1.3) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
10	0.11653467	0.116534674	0.883465326	0.11653467
11	0.09448826	0.211022934	0.788977066	0.10695186
⋮	⋮	⋮	⋮	⋮
20	0.02495953	0.618833336	0.381166664	0.06145757
⋮	⋮	⋮	⋮	⋮
50	0.00313637	0.879729602	0.120270398	0.02541488
⋮	⋮	⋮	⋮	⋮
100	0.00064413	0.950525416	0.049474584	0.01285213
⋮	⋮	⋮	⋮	⋮
1000	0.00000326	0.997491382	0.002508618	0.00129851
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

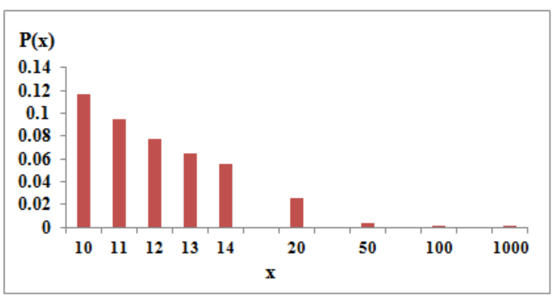

Fig. 5: The probability mass function of the EGW-III (0, 10. -1.3) distribution

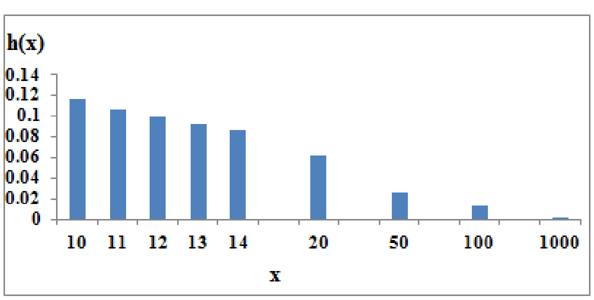

Fig. 6: The hazard function of the EGW-III (0, 10. -1.3) distribution

4.4. Table 4 displays the probability mass function [$P(x)$], the distribution function [$F(x)$], the survival function [$S(x)$], and the hazard function[$h(x)$] of the EGW-III (3, 9, -3.4) distribution, whereas, figures 7 and 8 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{(x-3)^{-3.4}-(x-2)^{-3.4}}{(9)^{-3.4}}, \qquad x=12,13,\cdots$$

Table 4: The EGW-III (0, 10, - 1.3) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
12	0.30108483	0.30108483	0.6989152	0.3010847
13	0.19345231	0.49453713	0.5054629	0.2767893
14	0.05539245	0.40968834	0.5903117	0.0857861
15	0.04750643	0.45719477	0.5428052	0.0804769
⋮	⋮	⋮	⋮	⋮
50	0.00313637	0.87972960	0.1202704	0.0254149
⋮	⋮	⋮	⋮	⋮
100	0.00064413	0.95052542	0.0494746	0.0128521
⋮	⋮	⋮	⋮	⋮
500	0.00000001	0.99999916	0.0000008	0.0068109
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

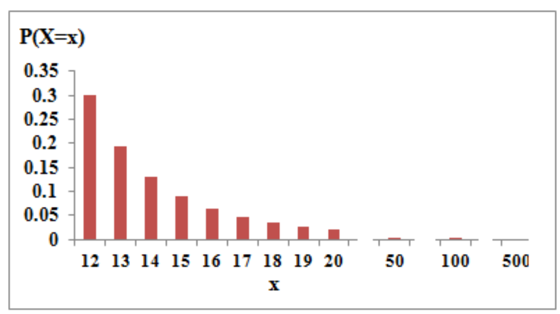

Fig. 7: The probability mass function of the EGW-III (3, 9. -3.4) distribution

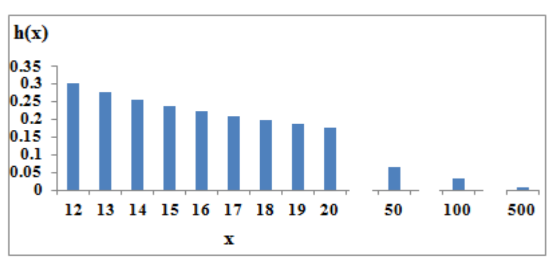

Fig. 8: The hazard function of the EGW-III (3, 9. -3.4) distribution

4.5. Table 5 displays the probability mass function [$P(x)$], the distribution function [$F(x)$], the survival function [$S(x)$], and the hazard function[$h(x)$] of the EGW-III (-5, 8, -2.5) distribution, whereas, figures 9 and 10 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{(x+5)^{-2.5}-(x+6)^{-2.5}}{(8)^{-2.5}}, \quad x=3,4,\cdots$$

Table 5: The EGW-III (-5, 8, -2.5) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
3	0.2550644	0.255064448	0.74493555	0.25506446
4	0.1725021	0.427566577	0.57243342	0.23156653
⋮	⋮	⋮	⋮	⋮
30	0.0016986	0.976720716	0.02327928	0.06800441
⋮	⋮	⋮	⋮	⋮
50	0.0003554	0.99228639	0.0077136	0.04404675
⋮	⋮	⋮	⋮	⋮
100	0.0000375	0.998435146	0.0015649	0.02341829
⋮	⋮	⋮	⋮	⋮
500	0.0000156	0.99996852	0.00003148	0.00493339
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

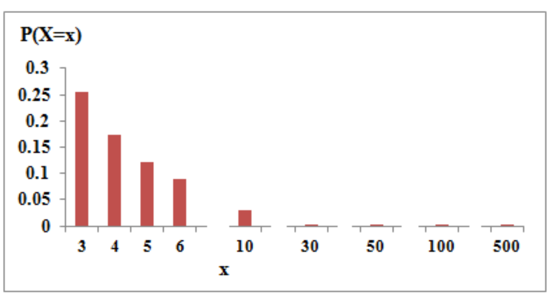

Fig. 9: The probability mass function of the EGW-III (-5, 8. -2.5) distribution

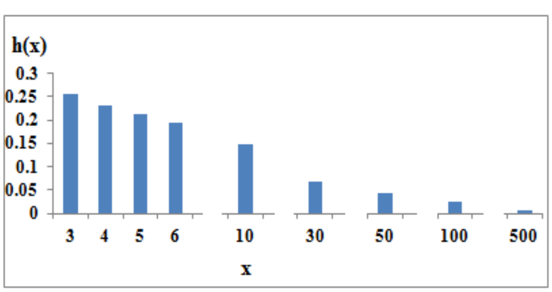

Fig. 10: The hazard function of the EGW-III (-5, 8. -2.5) distribution

4.6. Table 6 displays the probability mass function [$P(x)$], the distribution function [$F(x)$], the survival function [$S(x)$], and the hazard function[$h(x)$] of the EGW-III (8, 2, - 0.5) distribution, whereas, figures 11 and 12 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{(x-8)^{-0.5}-(x-7)^{-0.5}}{(2)^{-0.5}}$$

$$=\sqrt{2}[\frac{1}{\sqrt{x-8}}-\frac{1}{\sqrt{x-7}}], \quad x=10,11,\cdots$$

Table 6: The EGW-III (8, 2, -0.5) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
10	0.18350342	0.18350342	0.81649658	0.1835034
11	0.10938980	0.29289322	0.70710678	0.1339746
⋮	⋮	⋮	⋮	⋮
30	0.00419864	0.74599975	0.25400025	0.0162614
⋮	⋮	⋮	⋮	⋮
500	0.00006470	0.93630703	0.06369298	0.0010147
⋮	⋮	⋮	⋮	⋮
1000	0.00002261	0.9551213	0.04487871	0.0005037
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

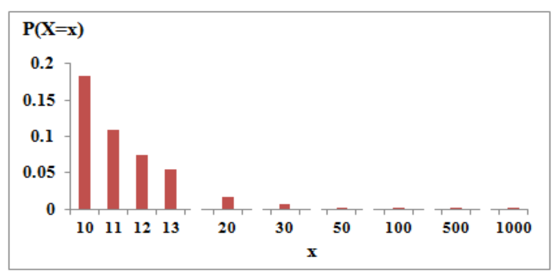

Fig. 11: The probability mass function of the EGW-III (8, 2, -0.5) distribution

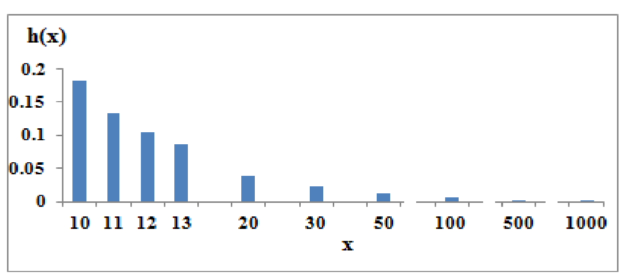

Fig. 12: The hazard function of the EGW-III (8, 2, -0.5) distribution

4.7. Table 7 displays the probability mass function $[P(x)]$, the distribution function $[F(x)]$, the survival function $[S(x)]$, and the hazard function$[h(x)]$ of the EGW-III (-8, 5, -1) distribution, whereas, figures 13 and 14 illustrate the probability mass function [P(x)] and hazard function [h(x)] of the same distribution, respectively.

$$P(X=x)=\frac{(x+8)^{-1}-(x+9)^{-1}}{(5)^{-1}}$$
$$=5[(x+8)^{-1}-(x+9)^{-1}],\quad x=-3,-2,\cdots$$

Table 7: The EGW-III (-8, 5, -1.0) distribution

x	$P(x)$	$F(x)$	$S(x)$	$h(x)$
-3	0.1666667	0.1666667	0.8333333	0.1666667
-2	0.1190472	0.2857142 9	0.7142857 1	0.1428571
-1	0.0892851	0.375	0.625	0.125
0	0.0694444	0.4444444	0.5555555 6	0.1111111
⋮	⋮	⋮	⋮	⋮
50	0.0014613	0.9152542	0.0847457 6	0.0169491 5
⋮	⋮	⋮	⋮	⋮
100	0.0004244	0.9541284	0.0458715 6	0.0091743 1
⋮	⋮	⋮	⋮	⋮
500	0.0000194	0.9901768	0.0098231 8	0.0019646 4
⋮	⋮	⋮	⋮	⋮
1000	0.0000042	0.9950446	0.0049554 0	0.0009911
⋮	⋮	⋮	⋮	⋮
∞	$\to 0$	$\to 1$	$\to 0$	
$\sum$	1			

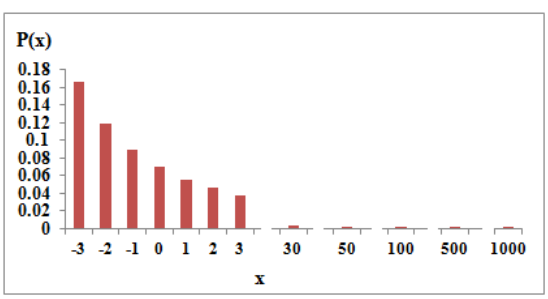

Fig. 13: The probability mass function of the EGW-III (-8, 5, -1.0) distribution

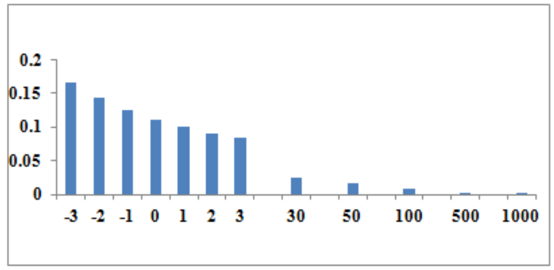

Fig. 14: The hazard function of the EGW-III (-8, 5, -1.0) distribution

The plots of the probability mass functions given by figures 1, 3, 5, 7, 9, 11, and 13 show a decreasing shape of the function whatever the values of a, s, and θ are considered, justifying that the mode of the EGW-III distribution is given as in equation (10).

The plots given in Figures 2, 4, 6, 8, 10, 12, and 14 also show a decreasing shape of the hazard function of the EGW-III distribution whatever the values of a, s, and θ are considered.

7. Real data

The EGW(III) distribution seems to be applicable for some real data such as:

7.1. Word Frequencies in Texts (Zipf's Law), as in linguistics, the frequency of words follows a power-law distribution: the n^{th} most frequent word has a frequency roughly proportional to $1/n$. If we model word ranks shifted by some a and scaled by s, the EGW(III) probability mass function could describe word occurrences beyond a certain threshold.

7.2. City Populations (Zipf's Law for Cities), as the population of cities often follows a power-law distribution, where the probability of a city having population x is roughly x^{-k}. The EGW(III) distribution could model city sizes beyond a certain minimum size $a+s$.

7.3. File Sizes in Computer Systems, as file size distributions in large storage systems often exhibit heavy-tailed behavior. If x represents file size, the EGW(III) distribution could describe the distribution of files larger than a certain threshold.

7.4. Earthquake Magnitudes (Truncated Gutenberg-Richter Law), as earthquake magnitudes follow an exponential distribution in energy, which translates to a power law in amplitude. If x is the magnitude (shifted and scaled appropriately), the EGW(III) distribution could describe tail behavior.

7.5. Wealth or Income Distributions (Pareto Tail), as the upper tail of income/wealth distributions often follows a Pareto distribution $P(X > x) \sim x^{-k}$. The EGW(III) distribution could model discrete wealth levels beyond a certain threshold.

8. Discussion and Conclusion

The EGW-III distribution was illustrated as a new semi-finite discrete probability distribution. The distribution exhibits heavy-tailed, power-law behavior, particularly for large values of s. The properties of the distribution were discussed. The first two moments and the variance of the distribution were found out and then approximated. The mode of the distribution was found to be the smallest value in the support of the distribution ($a + s$). Some examples considering different values of the parameters of the distribution were given. Finally, some kinds of real data for which the distribution may be applicable were also given.

9. Future work

- A simulation study may be done to generate data from the EGW-III distribution considering different values of the parameters of the distribution and the shape parameter may be then estimated using some methods of estimation.
- Another semi-finite discrete distribution is under consideration by putting $s < 0$ and $\theta < 0$ in the probability mass function (1).

Conflict of interest: The author certifies that there are no conflicts of interest.

References

[1.] Muiftah, M. S. A.(2018). On Expressing Continuous Distributions With Discrete Distributions. A PhD Thesis, FSSR, Cairo University, Egypt.

[2.] Muiftah, M. S. A.(2023). The Egwaider Type-I Distribution, Scientific Journal of Benghazi University, 36(2).

[3.] Muiftah, M. S. A.(2023). The Egwaider Type-II Distribution, International Journal of Scientific Research(IJSR), 2(11).

[4.] Crooks, G. E. (2017). Field guide to continuous probability distributions. http:// threeplusone.com/ field guide. v 0.11. Accessed on 4/7/2017.

[5.] Newman, M. E. J. (2005). "Power laws, Pareto distribution and Zipf"s law". Cotemporary Physics, 64(5), 323-351. arXiv:cond-mat/0412004.

[6.] Claust, A., Shalizi, C. R. & Newman, M. E. J. (2009). "Power law distributions in empirical data. SIAM Review, 51(4), 661-703. DOI:10.1137/070710111.