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The Extended Cone b_2-Metric-like Spaces on Banach Algebra with some Applications

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A B S T R A C T

In this paper, we initiate a groundbreaking investigation into the field of metric spaces by presenting the idea of an extended cone b_2-metric-like space within the context of Banach algebra., aiming to bridge and extend their properties. Within the framework of this newly defined space, we formulate multiple fixed point theorems., providing a foundation for further analytical investigations. To demonstrate the practical utility and applicability of our theoretical findings, we present an application to Fredholm integral equation, showcasing how the developed fixed-point results can be employed to solve real-world problems. Notably, our results contribute to the broader landscape of fixed-point theory and serve to generalize certain previously established results by Fernandez, J., et al., released in 2017 and 2022, thus advancing the field and offering more comprehensive tools for analysis.

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Introduction

The study of metric spaces forms a cornerstone of mathematical analysis, providing a framework for understanding notions of distance, convergence, and continuity. Over the years, mathematicians have continually sought to generalize these concepts, leading to a rich landscape of abstract spaces. This article explores the evolution of one such generalization, tracing the path from cone metric spaces to the newly introduced extended cone b_2-metric-like spaces over Banach algebras.

The expedition commenced in 2007 with the pioneering research of Huang and Zhang, who introduced the notion of cone metric spaces [13]. This advancement substituted the traditional set of real numbers in the definition of metric spaces with elements derived from an ordered Banach space. This seemingly subtle change opened doors to new analytical techniques and spurred further research, leading to numerous fixed-point results (see [1, 2, 8]). Building on this foundation, Singh et al. in 2012 introduced cone 2-metric spaces [26], another generalization aimed at capturing more complex relationships between points. Again, this space provided a fertile ground for establishing fixed point theorems.

A significant advancement arrived in 2013, courtesy of Liu and Xu [19], with the introduction of cone metric spaces on Banach algebras. This further abstract setting allowed for a wider range of applications and provided a springboard for

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even more general spaces. The quest for generalization continued, leading Zead Mustafa in 2014 to define b_2-metric spaces [28]. This novel form of generalized metric space acts as an extension of both 2-MS and b-MS, offering a unified framework for studying these structures.

2017 saw the introduction of cone b_2-metric-like spaces over Banach algebras with a non-normal case by Fernandez et al. [6] and subsequently the concept of cone $\boldsymbol{b_2}$-metric spaces over Banach algebras [7], representing significant steps towards even greater abstraction and flexibility. The exploration did not stop there. Recognizing the limitations of constant coefficients in these spaces, Elmabrok and Alkaleeli proposed in 2018 the extended b_2-metric space [4]. They cleverly substituted a function for the constant coefficient in the $\boldsymbol{b_2}$-metric inequality, allowing for a more nuanced representation of the distance relationship. Research pertaining to extended b_2-metric spaces is documented in references [3-5]. More recently, in 2022, Fernandez et al. [10] investigated extended cone b-metric spaces and derived unique fixed-point results, pushing the boundaries of these generalized spaces.

Inspired by these developments, a new paper introduces the extended cone b_2-metric-like space over Banach algebra. This novel space builds upon the previous work on cone b_2-metric spaces [7] and extended cone b-metric spaces [10], offering a powerful generalization. The key innovation lies in replacing the constant $s \geq 1$ in the rectangle inequality with a function $\theta(\xi, \eta, \zeta)$, providing even greater flexibility in modeling distances. The paper goes on to establish vital fixed point theorems, proving both existence and uniqueness results within this new space. Furthermore, illustrative examples are provided to demonstrate the practical applicability and utility of the newly derived theorems.

The sets of real numbers, positive real numbers, and positive integers are denoted in the supplement by the symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively. We will also define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

1. Preliminaries

This part starts by revisiting fundamental definitions and concepts that will be instrumental in our subsequent discussions. Notably, Huang and Zhang [13] pioneered the introduction of the concept of a cone metric space. Building upon this foundation, Huang and Radenović [14] further generalized the notion by introducing cone metric spaces over Banach algebras. Specifically, we will consistently refer to $\mathcal{A}$ as a real Banach algebra, which is equipped with a multiplication operation, satisfying specific properties. These properties, applicable to all $\xi, \eta, \mu \in \mathcal{A},\ \alpha \in \mathbb{R}$:

1) $(\xi\eta)\mu = \xi(\eta\mu)$,
2) $\xi(\eta + \mu) = \xi\eta + \xi\mu$ and $(\xi + \eta)\mu = \xi\mu + \eta\mu$,
3) $\alpha(\xi\eta) = (\alpha\xi)\eta = \xi(\alpha\eta)$,
4) $\|\xi\eta\| \leq \|\xi\| \cdot \|\eta\|$.

If $e_{\mathcal{A}}\xi = \xi e_{\mathcal{A}} = \xi$ for all $\xi \in \mathcal{A}$, then $e_{\mathcal{A}} \in \mathcal{A}$ is called unit. If there exists a non-zero element $\xi \in \mathcal{A}$, such that $\xi\eta = \eta\xi = e_{\mathcal{A}}$, then $\xi \in \mathcal{A}$, is said to be invertible, ξ^{-1} is the inverse of ξ. Further details can be found in [22].

Proposition 2.1 [24]. Consider $\mathcal{A}$ to be a real Banach algebra equipped with a unit $e_{\mathcal{A}}$, where the spectral radius of an element ξ in $\mathcal{A}$ is less than 1, that is,

$$\rho(\xi) = \lim_{n\to\infty} \|\xi^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|\xi^n\|^{\frac{1}{n}} < 1,$$

then $(e_{\mathcal{A}} - \xi)$ is invertible and $(e_{\mathcal{A}} - \xi)^{-1} = \sum_{i=0}^{\infty} \xi^i$.

Remark 2.2 [24]

Where $\mathcal{A}$ represent a real Banach algebra that includes a unit element.$e_{\mathcal{A}}$, the spectral radius $\rho(\xi)$ of $\xi \in \mathcal{A}$ adheres $\rho(\xi) \leq \|\xi\|$.

Remark 2.3 [30].

If $\rho(\xi) < 1$ then $\|\xi\|^n \to 0$ $(n \to \infty)$.

Definition 2.4 [24] A non-empty subset $\mathbb{P}$ of a unital Banach algebra $\mathcal{A}$ is referred to as a cone if

1) $\mathbb{P}$ is closed and $\{\theta_{\mathcal{A}},\ e_{\mathcal{A}}\} \subset \mathbb{P}$,
2) $\alpha\,\xi + \beta\,\eta \subset \mathbb{P}$, for all $\xi, \eta \in \mathbb{P}$ and $\alpha, \beta \in \mathbb{R}^+$,
3) $\mathbb{P}^2 = \mathbb{P}\mathbb{P} \subset \mathbb{P}$,
4) $\mathbb{P} \cap (-\mathbb{P}) = \{\theta\}$,

where $\theta_{\mathcal{A}}$ represents the zero element of the Banach algebra $\mathcal{A}$.

Regarding a specific cone $\mathbb{P} \subset \mathcal{A}$. A partial ordering $\precsim$ can be established in relation to $\mathbb{P}$ such that $\xi \precsim \eta$ iff $(\eta - \xi) \in \mathbb{P}$. $\xi \prec \eta$ will represent $\xi \precsim \eta$ and $\xi \neq \eta$, while $\xi \ll \eta$ will stand for $(\eta - \xi) \in \operatorname{int}\mathbb{P}$, where $\operatorname{int}\mathbb{P}$ represents the interior of $\mathbb{P}$. If $\operatorname{int}\mathbb{P} \neq \emptyset$, in this case, $\mathbb{P}$ is referred to as a solid cone. The cone $\mathbb{P}$ is referred to as normal if there exists a number $M > 0$, such that for every instance $\xi, \eta \in \mathcal{A}$,

$$\theta_{\mathcal{A}} \precsim \xi \precsim \eta \Longrightarrow \|\xi\| \leq M\|\eta\|,$$

Where the positive value M that fulfills the normality requirement is referred to as the normal constant of $\mathbb{P}$.

Definition 2.5 [25]: Suppose that $\mathbb{P}$ represent a solid cone within a Banach algebra . A sequence $\{v_n\} \subset \mathbb{P}$ is classified as a c-sequence if for every $\theta_{\mathcal{A}} \ll c$, there is a natural number N such that $v_n \ll c$ for all $n > N$.

Lemma 2.6 [30] Let $\mathbb{P}$ represent a solid cone within a Banach algebra $\mathcal{A}$. Assume that l is an element of $\mathbb{P}$ and $\{v_n\}$ constitutes a c-sequence in $\mathbb{P}$. Consequently, $\{lv_n\}$ will also form c-sequence.

In the sections that follow, we will consistently operate under the presumptions that $\mathcal{A}$ represents a real unital Banach algebra characterized by a solid cone $\mathbb{P}$. Additionally, we will use the following notations:

a. CMS as cone-metric space,
b. ECMS as extended cone-metric space,
c. b_2MS as b_2-metric space,
d. Cb_2MSBA as cone b_2-metric space over Banach algebra,
e. ECb_2MSBA as extended cone b_2-metric space over Banach algebra,

Definition 2.7 [19] Let X be a set that is not empty. It is claimed that the mapping $d_c: X \times X \longrightarrow \mathcal{A}$ is reported to be a CMBA on X if for each $\xi, \eta, \zeta \in X$, the subsequent circumstances are met:

1) $d_c(\xi, \eta) = \theta_{\mathcal{A}}$, iff $\xi = \eta$,
2) $d_c(\xi, \eta) = d_c(\eta, \xi)$,
3) $d_c(\xi, \zeta) \precsim [d_c(\xi, \eta) + d_c(\eta, \zeta)]$.

A CMSBA is the space (X, d_c).

Definition 2.8 [15]: Let X be a nonempty set and $s \geq 1$ be a conctant. A mapping $d_{cb}: X \times X \longrightarrow \mathcal{A}$, where A is a Banach algebra, is regarded as a CbMBA on X if for every $\xi, \eta, \zeta \in X$, the following circumstances are true:

1) $d_{cb}(\xi, \eta) = \theta_{\mathcal{A}}$, iff $\xi = \eta$,
2) $d_{cb}(\xi, \eta) = d_{cb}(\xi, \eta)$,
3) $d_{cb}(\xi, \zeta) \precsim s[d_{cb}(\xi, \eta) + d_{cb}(\eta, \zeta)]$.

These conditions ensure that the mapping d_{cb} behaves in a way that is analogous to a regular metric, but with values residing in the ordered Banach algebra $\mathcal{A}$ rather than the real numbers. A CbMSBA is the space (X, d_{cb}).

Definition 2.9 [18].**:** Let X be a nonempty set and $\theta: X \times X \to [\,1, \infty)$ be a mapping. A mapping $d_{ecb}: X \times$

$X \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra, is reported to be an ECbMBA on X if for all $\xi,\eta,\zeta \in X$, with the assumption that the following conditions are met:

1) $d_{ecb}(\xi,\eta) = \theta_{\mathcal{A}}$, iff $\xi = \eta$,

2) $d_{ecb}(\xi,\eta) = d_{ecb}(\eta,\xi)$,

3) $d_{ecb}(\xi,\zeta) \precsim \theta(\xi,\eta)[d_{ecb}(\xi,\eta) + d_{ecb}(\eta,\zeta)]$.

The triplet $(X, d_{ecb}, \mathcal{A})$ is referred as an ECbMSBA.

Definition 2.10 [29] Let X be a nonempty set, $s \geq 1$ be a real number. A map $d_{b2}: X \times X \times X \longrightarrow \mathbb{R}$ is referred as a b_2M on X, if it meets these stipulated requirements:

1) For each couple of points $\xi,\eta \in X$, there is a point $\zeta \in X$ such that $d_{b_2}(\xi,\eta,\zeta) \neq 0$,

2) If at least two of three points ξ, η, ζ are the same, then $d_{b_2}(\xi,\eta,\zeta) = 0$,

3) The symmetry, for all ξ, η, $\zeta \in X$

$$d_{b_2}(\xi,\eta,\zeta) = d_{b_2}(\xi,\zeta,\eta) = d_{b_2}(\eta,\xi,\zeta) = d_{b_2}(\eta,\zeta,\xi) = d_{b_2}(\zeta,\xi,\eta) = d_{b_2}(\zeta,\eta,\xi),$$

4) The rectangle inequality, for all $\xi,\eta,\zeta,\varrho \in X$

$$d_{b_2}(\xi,\eta,\zeta) \leq s[d_{b_2}(\xi,\eta,\varrho) + d_{b_2}(\eta,\zeta,\varrho) + d_{b_2}(\zeta,\xi,\varrho)].$$

The structure $\left(X, d_{b_2}\right)$ is a b_2MS.

Definition 2.11 [7] Let X be a nonempty set, $s \geq 1$ be a real number. A mapping $d_{cb_2}: X \times X \times X \longrightarrow \mathcal{A}$ witch meet the requirements listed below:

1) Each couple on different points $\xi,\eta \in X$, had a point $\zeta \in X$, such that $d_{cb_2}(\xi,\eta,\zeta) \neq 0_{\mathcal{A}}$,

2) If at least two of three points ξ,η,ζ are the same, then $d_{cb_2}(\xi,\eta,\zeta) = 0_{\mathcal{A}}$,

3) The symmetry, for all $\xi,\eta,\zeta \in X$,

$$d_{cb_2}(\xi,\eta,\zeta) = d_{cb_2}(\xi,\zeta,\eta) = d_{cb_2}(\eta,\xi,\zeta) = d_{cb_2}(\eta,\zeta,\xi) = d_{cb_2}(\zeta,\xi,\eta) = d_{cb_2}(\zeta,\eta,\xi),$$

4) The rectangle inequality, for all $\xi,\eta,\zeta,\varrho \in X$

$$d_{cb_2}(\xi,\eta,\zeta) \precsim s\begin{bmatrix} d_{cb_2}(\xi,\eta,a) \\ +d_{cb_2}(\eta,\zeta,a) \\ +d_{cb_2}(\zeta,\xi,a) \end{bmatrix}.$$

The triplet $(X, d_{cb2}, \mathcal{A})$ is indicated as Cb_2MSBA.

2. Extended cone b_2-metric-like space over Banach algebra

In this section, we embark on the introduction of a novel and generalized metric space, christened the ECb_2MSBA. This new structure arises as a significant extension, encompassing and unifying the properties of both Cb_2MS, as comprehensively explored in reference [7] and ECbMSBA as detailed in [10]. By forging this connection, we aim to create a more versatile framework capable of addressing a wider range of problems and offering deeper insights into fixed-point theory and related mathematical areas within the context of Banach algebras. This generalized space allows for more flexible distance measurements, enriching the potential applications across disciplines relying on metric space analysis.

Definition 3.1. Consider X as a nonempty set, and let $\mathcal{A}$ represent a real unital Banach algebra equipped with a solid cone $\mathbb{p}$ and $\theta: X \times X \times X \to [1,\infty)$ be a mapping. A mapping $d_{ecb_2}: X \times X \times X \longrightarrow [0,\infty)$ is referred as ECb_2MBA on X if for all $\xi,\eta,\zeta \in X$, the following circumstances are true:

1) For any two different points ξ and η in the set X, there exists a point ζ also in X, such that $d_{ecb_2}(\xi,\eta,\zeta) \neq \theta_{\mathcal{A}}$,

2) If at least two of three points ξ,η,ζ are the same, then $d_{ecb_2}(\xi,\eta,\zeta)=\theta_{\mathcal{A}}$.

3) $d_{ecb_2}(\xi,\eta,\zeta)=d_{ecb2}(p\{\xi,\zeta,\eta\})$, where p is a permutation of ξ,η,ζ (Symmetry),

4) $d_{ecb_2}(\xi,\eta,\zeta)\precsim\theta(\xi,\eta,\zeta)\begin{bmatrix} d_{ecb_2}(\xi,\eta,\varrho)\\ +d_{ecb_2}(\eta,\zeta,\varrho)\\ +d_{ecb_2}(\zeta,\xi,\varrho)\end{bmatrix}$ for all

$\xi,\eta,\zeta,\varrho\in X$ (Rectangle inequality).

The triplet $(X,d_{ecb2},\mathcal{A})$ is indicated as ECb_2MSBA.

Note, that the mapping d_{ecb2} provides a generalization of traditional metric spaces, allowing distances to be elements within the solid cone $\mathbb{p}$ of a Banach algebra $\mathcal{A}$, rather than just non-negative real numbers. The presence of $\theta(\xi,\eta,\zeta)$ allows for the coefficient in the triangle inequality to vary depending on the points involved, adding a layer of flexibility compared to standard Cb_2M.

Remarks 3.2

- **i.** An ECb_2MSBA is more comprehensive than a standard Cb_2MSBA. This is evident as, if $\theta(\xi,\eta,\zeta)=s>1$ for all $\xi,\eta,\zeta\in X$, it results in a Cb_2MSBA.
- **ii.** If $\theta(\xi,\eta,\zeta)=1$ for all, $\xi,\eta,\zeta\in X$, then an ECb_2MSBA reduces to a CMSBA.
- **iii.** A b_2MS is an ECb_2MSBA, where $\theta(\xi,\eta,\zeta)=s>1$ for all $\xi,\eta,\zeta\in X$ and $\mathcal{A}=\mathbb{R}$ with the cone $\mathbb{p}=[0,\infty)$,
- **iv.** If $\theta(\xi,\eta,\zeta)=1$ for all $\xi,\eta,\zeta\in X$ and $\mathcal{A}=\mathbb{R}$ with the cone $\mathbb{p}=[0,\infty)$, then an ECb_2MSBA simplifies to an usual 2-MS.
- **v.** If $\mathcal{A}=\mathbb{R}$ with the cone $\mathbb{p}=[0,\infty)$, then an ECb_2MSBA reduces to an Eb_2MS,
- **vi.** Using condition (1) in Definition 3.1, it easily confirmed it for every $\varrho\in X$, $d(\xi,\eta,\varrho)=\theta_{\mathcal{A}}$, then $\xi=\eta$.

Example 3.3: Assume that $\mathcal{A}=C[0,\ 1]$ is a collection of continuous functions on the $[0,1]$ interval with the norm $\|\xi\|=\|\xi\|_\infty+\|\acute{\xi}\|_\infty$ and the standard definition of multiplication. Consequently, $\mathcal{A}$ constitutes a Banach algebra that includes a unit element $e_{\mathcal{A}}$.

Set $\mathbb{p}=\{\xi\in\mathcal{A}:\xi(t)\geq 0,\ t\in[a,b]\}$ and $X=[0,1]$.

Define $\theta: X\times X\times X\longrightarrow[1,\infty)$ by

$$\theta(\xi,\eta,\zeta)=\begin{cases}|\xi|+|\eta|+|\zeta| & \text{for } \xi\neq\eta\neq\zeta\,,\\ 1, & \text{otherwise.}\end{cases}$$

Create a mapping $d_{ecb_2}: X\times X\times X\to\mathcal{A}$ by

$$d_{ecb_2}(\xi,\eta,\zeta)(t)=\begin{cases}\left[\min\begin{Bmatrix}|\xi-\eta|,\\ |\xi-\zeta|,\\ |\eta-\zeta|\end{Bmatrix}\right]^p e^t, & \xi\neq\eta\neq\zeta\neq\xi\\ 0, & \text{otherwise}\end{cases}.$$

For $p>1$ and for every $t\in[0,1]$, it is evident that conditions (1) through (3) are fulfilled. Now, to prove (4) we take $\xi,\eta,\zeta,\varrho\in X$ as arbitrary, using convexity of the function $f(\xi)=\xi^p$, on $[0,1]$, then by Jensen inequality, we see that

i. If $\xi=\eta$ or $=\zeta$, then (4) is clear.

ii. If $\xi\neq\eta\neq\zeta\neq\varrho$, then

$$d_{ecb_2}(\xi,\eta,\zeta)(t)=\begin{cases}\left[\min\begin{Bmatrix}|\xi-\eta|,\\ |\xi-\zeta|,\\ |\eta-\zeta|\end{Bmatrix}\right]^p e^t, & \xi\neq\eta\neq\zeta\neq\xi\\ 0, & \text{otherwise}\end{cases},$$

$$\precsim(1+|\xi|+|\eta|+|\zeta|)\left[\min\begin{Bmatrix}|\xi-\eta|,\\ |\xi-\zeta|,\\ |\eta-\zeta|\end{Bmatrix}\right]^p e^t,$$

$$\precsim (1+|\xi|+|\eta|+|\zeta|)\left[\begin{pmatrix} \min\begin{Bmatrix}|\xi-\eta|,\\ |\xi-\varrho|,\\ |\eta-\varrho|\end{Bmatrix} \\ +\min\begin{Bmatrix}|\varrho-\eta|,\\ |\varrho-\zeta|,\\ |\eta-\zeta|\end{Bmatrix} \\ +\min\begin{Bmatrix}|\xi-\varrho|,\\ |\xi-\zeta|,\\ |\varrho-\zeta|\end{Bmatrix}\end{pmatrix}\right]^p e^t,$$

$$\precsim (1+|\xi|+|\eta|+|\zeta|)\begin{bmatrix} \left(\min\begin{Bmatrix}|\xi-\eta|,\\ |\xi-\varrho|,\\ |\eta-\varrho|\end{Bmatrix}\right)^p e^t \\ +\left(\min\begin{Bmatrix}|\varrho-\eta|,\\ |\varrho-\zeta|,\\ |\eta-\zeta|\end{Bmatrix}\right)^p e^t \\ +\left(\min\begin{Bmatrix}|\xi-\varrho|,\\ |\xi-\zeta|,\\ |\varrho-\zeta|\end{Bmatrix}\right)^p e^t\end{bmatrix},$$

$$=\theta(\xi,\eta,\zeta)\begin{bmatrix} d_{ecb_2}(\xi,\eta,\varrho)(t) \\ +d_{ecb_2}(\eta,\zeta,\varrho)(t) \\ +d_{ecb_2}(\zeta,\xi,\varrho)(t)\end{bmatrix},$$

$$d_{ecb_2}(\xi,\eta,\zeta)(t)\precsim\theta(\xi,\eta,\zeta)\begin{bmatrix} d_{ecb_2}(\xi,\eta,\varrho)(t) \\ +d_{ecb_2}(\eta,\zeta,\varrho)(t) \\ +d_{ecb_2}(\zeta,\xi,\varrho)(t)\end{bmatrix}.$$

Thus, the rectangle inequality holds. Then $(X,\ d_{ecb_2},\ \mathcal{A})$ is an ECb_2MSBA.

Example 3.4 Consider $\mathcal{A}=C^1_{\mathbb{R}}[0,1]$ as the set comprising all differentiable functions with real values that possess a continuous derivative on the interval [0,1], in accordance with the norm $\|\xi\|=\|\xi\|_\infty+\|\acute{\xi}\|_\infty$. Establish the concept of multiplication in $\mathcal{A}$ with exact multiplication of points. Then $\mathcal{A}$ is a Banach algebra that is real and has a unit $e_{\mathcal{A}}=1$. Set $\mathbb{p}=\{\xi\in\mathcal{A}:\xi(t)\geq 0,\ t\in[0,1]\}$ is a cone in $\mathcal{A}$ and $=[0,1]$.

Define $\theta: X\times X\times X\longrightarrow[1,\infty)$ by

$$\theta(\xi,\eta,\zeta)=\begin{cases}1+|\xi|+|\eta|, & \text{for } \xi\neq\eta\,,\\ 1+|\xi|+|\zeta|, & \text{for } \xi\neq\zeta,\\ 1, & \text{otherwise.}\end{cases}$$

Moreover, establish a mapping $d_{ecb_2}: X\times X\times X\to\mathcal{A}$ by

$$d_{ecb_2}(\xi,\eta,\zeta)(t)=\begin{cases}[\xi\eta+\eta\zeta+\zeta\xi]^p e^t, & \xi\neq\eta\neq\zeta\neq\xi\\ 0, & \text{otherwise}\end{cases}.$$

For all $,\eta,\zeta\in X$, where $p>1$, (see [7]). Clearly, it's the convexity of the function $f(\xi)=\xi^p$, for $\xi\geq 0$, according to Jensen's inequality, we can conclude that

$$(\xi+\eta+\zeta)^p\leq 3^{p-1}(\xi^p+\eta^p+\zeta^p).$$

Then, we obtain the result that $(X,\ d_{ecb_2},\ \mathcal{A})$ is an ECb_2MSBA, with $\theta(\xi,\eta,\zeta)\leq 3^{p-1}$.

Definition 3.5 Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence in an ECb_2MSBA $(X,\ d_{ecb_2},\ \mathcal{A})$:

1) A sequence $\{\xi_n\}$ is considered convergent to $\xi\in X$, for each $c\gg\theta_{\mathcal{A}}$ there is a natural number N such that $d_{ecb_2}(\xi_n,\xi,\varrho)\ll c$ for all $n\geq N$. Written as

$$\lim_{n\to\infty}\xi_n=\xi \text{ or } \xi_n\to\xi\ (n\to\infty).$$

2) It is argued that a sequence $\{\xi_n\}$ is Cauchy sequence whenever for every $c\gg\theta_{\mathcal{A}}$ ther exists a natural number N such that $d_{ecb_2}(\xi_n,\xi_m,\varrho)\ll c$ for all $n,m\geq N$.

Definition 3.6 : An ECb_2MSBA $(X,\ d_{ecb_2},\ \mathcal{A})$ is considered complete if all of the Cauchy sequences $\{\xi_n\}$ converge to $\xi\in X$.

Definition 3.7: Let $(X,\ d_{ecb_2},\ \mathcal{A})$ be an ECb_2MSBA. The extended b_2-metric d_{ecb2} is referred to as continuous if

$$d_{ecb_2}(\xi_n, \xi, \varrho) \ll c \text{ and } d_{ecb_2}(\eta_n, \eta, \varrho) \ll c$$

implies that

$$d_{ecb_2}(\xi_n, \eta_n, \varrho) \longrightarrow d_{ecb_2}(\xi, \eta, \varrho),$$

for all sequence $\{\xi_n\}, \{\eta_n\}$ in $X, n \geq N$ and $\xi, \eta, \varrho \in X$.

1. Fixed point results of generalized Lipschitz mappings

This section outlines the generalized Lipschitz function within the ECb_2M framework of the Banach algebra $\mathcal{A}$.

Definition 4.1: A map $T: X \to X$ on an ECb_2MSBA is referred to as a generalized Lipschitz mapping in the event that a vector $\rho(l) < 1$ for all $\xi, \eta, \zeta \in X$, such that

$$d_{ecb_2}(T\xi, T\eta, \varrho) \preccurlyeq l\, d_{ecb_2}(\xi, \eta, \varrho).$$

Definition 4.2: A self-mapping T on an ECb_2MSBA $(X, d_{ecb_2}, \mathcal{A})$ is called d_{ecb_2}-orbitally continuous if for all $\xi, \eta, \varrho \in X$

$$d_{ecb2}(T^n\xi, \eta, \varrho) \to \theta_{\mathcal{A}}, \text{ as } n \to \infty,$$

implies that

$$d_{ecb_2}(TT^n\xi, T\eta, \varrho) \to \theta_{\mathcal{A}}, \text{ as } n \to \infty.$$

Definition 4.3 [13]: For a mapping $T: X \to X$ and an element $\xi_0 \in X$ for every $n \in \mathbb{N}$, the orbit of ξ_0 under T is represented by the subsequent sequences of points:

$$O(x_0) = \{\xi_0, \quad T\xi_0, \quad T^2\xi_0 \ \dots, T^n\xi_0, \dots\}.$$

Definition 4.4: Let T be a self- mapping on a set X that is not empty. An ECb_2MSBA $(X, d_{ecb_2}, \mathcal{A})$ is claimed to be T-orbitally complete, if each Cauchy sequence in $OT(\xi_0)$ converges in X, where $x_0 \in X$.

In the context of an ECb_2MSBA $(X, d_{ecb_2}, \mathcal{A})$, an analog of the Banach contraction principle is our initial theorem.

Lemma 4.5: Consider $\mathcal{A}$ as a real Banach algebra and

$$l\,(\neq \theta_{\mathcal{A}}) \in \mathbb{p}: \lim_{n\to\infty} \frac{\|l^{n+1}\|}{\|l^n\|} \text{ exists}$$

then this limit is equal to $\rho(l)$.

We shall define a subset of $\mathcal{A}$ in the following manner:

$$\mathbb{p}^* := \left\{ l\,(\neq \theta_{\mathcal{A}}) \in \mathbb{p}: \lim_{n\to\infty} \frac{\|l^{n+1}\|}{\|l^n\|} \text{ exists} \right\}.$$

Theorem 4.6 : Let $\mathbb{p}$ be a solid cone in $\mathcal{A}$. Let $(X, d_{ecb_2}, \mathcal{A})$ be a complete ECb_2MSBA, such that d_{ecb_2} is continuous. Suppose that $T: X \to X$ be a mapping satisfy, for all, $\xi, \eta, \varrho \in X$

$$d_{ecb_2}(T\xi, T\eta, \varrho) \preccurlyeq l\, d_{ecb_2}(\xi, \eta, \varrho), \qquad (4.1)$$

where $l \in \mathbb{p}^*$ be such that for each $\xi_0 \in X$,

$$\lim_{n,m\to\infty} \theta(\xi_n, \xi_m, \varrho) < \frac{1}{\rho(l)},$$

where $\{\xi_n\} = \{T^n\xi_0\}$, $n = 1,2, \dots$. Then T has exactly one fixed point u. Additionally, each $\eta \in X$, $T^n\eta \to u$.

Proof:

Select any $\xi_0 \in X$ as an arbitrary and Create the iterative sequence $\{\xi_n\}$ by

$$\xi_0, \ \xi_1 = T\xi_0, \ \xi_2 = T\xi_1 = T^2\xi_0, \dots, \ \xi_n = T^n\xi_0 \ \dots$$

If $\xi_n = \xi_{n+1}$ for some n, there is nothing to prove. Thus, let's assume that $\xi_n \neq \xi_{n+1}$ for each $n \in \mathbb{N}_0$. By the contractive condition (4.1) , we have

$$d_{ecb_2}(\xi_n, \xi_{n+1}, \varrho) = d_{ecb_2}(T^n\xi_0, T^{n+1}\xi_0, \varrho)$$

$$\preccurlyeq l^n d_{ecb_2}(\xi_0, \xi_1, \varrho). \qquad (4.2)$$

Let's now demonstrate that the sequence $\{\xi_n\}$ in X is Cauchy.

For $m > n$, we have

$$d_{ecb_2}(\xi_n,\xi_m,\varrho) \preccurlyeq \theta(\xi_n,\xi_m,\varrho)\begin{bmatrix} d_{ecb_2}(\xi_n,\xi_{n+1},\varrho) \\ +d_{ecb_2}(\xi_n,\xi_{n+1},\xi_m) \\ +d_{ecb_2}(\xi_{n+1},\xi_m,\varrho) \end{bmatrix}$$

$$\precsim \theta(\xi_n,\xi_m,\varrho)\begin{bmatrix} l^n d_{ecb_2}(\xi_0,\xi_1,\varrho) \\ + l^n d_{ecb_2}(\xi_0,\xi_1,\xi_m) \end{bmatrix} + \theta(\xi_n,\xi_m,\varrho)d_{ecb_2}(\xi_{n+1},\xi_m,\xi),$$

$$\precsim \theta(\xi_n,\xi_m,\varrho)\, l^n d_{ecb_2}(\xi_0,\xi_1,\varrho) + \theta(\xi_n,\xi_m,\varrho)\, l^n d_{ecb_2}(\xi_0,\xi_1,\xi_m)$$

$$+\theta(\xi_n,\xi_m,\varrho)\theta(\xi_{n+1},\xi_m,\varrho)\begin{bmatrix} d_{ecb_2}(\xi_{n+2},\xi_{n+1},\varrho) \\ +d_{ecb_2}(\xi_{n+2},\xi_{n+1},\xi_m) \\ +d_{ecb_2}(\xi_{n+2},\xi_m,\varrho) \end{bmatrix},$$

$$\precsim \begin{bmatrix} \theta(\xi_n,\xi_m,\varrho)l^n \\ +\theta(\xi_n,\xi_m,\varrho)\theta(\xi_{n+1},\xi_m,\varrho)l^{n+1} \end{bmatrix} d_{ecb_2}(\xi_0,\xi_1,\varrho)$$

$$+\begin{bmatrix} \theta(\xi_n,\xi_m,\varrho)l^n \\ +\theta(\xi_n,\xi_m,\varrho)\theta(\xi_{n+1},\xi_m,\varrho)l^{n+1} \end{bmatrix} d_{ecb2}(\xi_0,\xi_1,\xi_m)$$

$$+ \theta(\xi_n,\xi_m,\varrho)\,\theta(\xi_{n+1},\xi_m,\varrho)\, d_{ecb_2}(\xi_{n+2},\xi_m,\varrho),$$

$$\precsim \begin{bmatrix} \theta(\xi_n,\xi_m,\varrho)l^n \\ +\theta(\xi_n,\xi_m,\varrho)\theta(\xi_{n+1},\xi_m,\varrho)l^{n+1} \\ +\cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ +\theta(\xi_n,\xi_m,\varrho)\theta(\xi_{n+1},\xi_m,\varrho)\ldots \\ \theta(\xi_{m-2},\xi_m,\varrho)\theta(\xi_{m-1},\xi_m,\varrho)l^{m-1} \end{bmatrix}\begin{bmatrix} d_{ecb2}(\xi_0,\xi_1,\varrho) \\ +d_{ecb_2}(\xi_0,\xi_1,\xi_m) \end{bmatrix},$$

$$\precsim \left(\sum_{i=1}^{m-1} l^i \prod_{j=1}^{i} \theta(\xi_j,\xi_m,\varrho)\right)\begin{bmatrix} d_{ecb_2}(\xi_0,\xi_1,\varrho) \\ +d_{ecb_2}(\xi_0,\xi_1,\xi_m)\,. \end{bmatrix} \tag{4.3}$$

Also, we have

$$d_{ecb_2}(\xi_0,\xi_1,\xi_m) \preccurlyeq \theta(\xi_0,\xi_1,\xi_m)\begin{bmatrix} d_{ecb_2}(\xi_0,\xi_1,\xi_{m-1}) \\ +d_{ecb_2}(\xi_{m-1},\xi_m,\xi_0) \\ +d_{ecb_2}(\xi_{m-1},\xi_m,\xi_1) \end{bmatrix}$$

$$\preccurlyeq \theta(\xi_0,\xi_1,\xi_m)\begin{bmatrix} d_{ecb_2}(\xi_0,\xi_1,\xi_{m-1}) \\ + k^{m-1}d_{ecb_2}(\xi_0,\xi,\xi_0) \\ + k^{m-2}d_{ecb_2}(\xi_1,\xi_2,\xi_1) \end{bmatrix},$$

$$= \theta(\xi_0,\xi_1,\xi_m)d_{ecb_2}(\xi_0,\xi_1,\xi_{m-1})$$

$$\preccurlyeq \theta(\xi_0,\xi_1,\xi_m)\theta(\xi_0,\xi_1,\xi_{m-1})d_{ecb_2}(\xi_0,\xi_1,\xi_{m-2})$$

$$\preccurlyeq \ \ldots$$

In this way

$$d_{ecb_2}(\xi_0,\xi_1,\xi_m) \preccurlyeq \theta(\xi_0,\xi_1,\xi_m)\ \theta(\xi_0,\xi_1,\xi_{m-1}) \ldots \theta(\xi_0,\xi_1,\xi_2)d_{ecb_2}(\xi_0,\xi_1,\xi_1) = \theta_{\mathcal{A}}.$$

Hence $d_{ecb_2}(\xi_0,\xi_1,\xi_m) = \theta_{\mathcal{A}}$.

Therefore, equation (4.3), becomes

$$d_{ecb_2}(\xi_n,\xi_m,\varrho) \preccurlyeq \left(\sum_{i=1}^{m-1} l^i \prod_{j=1}^{i} \theta(\xi_j,\xi_m,\varrho)\right) d_{ecb_2}(\xi_0,\xi_1,\varrho). \tag{4.4}$$

Again we have,

$$\left\|\sum_{i=1}^{m-1} l^i \prod_{j=1}^{i} \theta(\xi_j,\xi_m,\varrho)\right\| \leq \sum_{i=1}^{m-1} \prod_{j=1}^{i} \theta(\xi_j,\xi_m,\varrho)\|l^i\|.$$

For, $i \in \mathbb{N}$, let us define

$$a_i^m = \prod_{j=1}^{i} \theta(\xi_j,\xi_m,\varrho)\,\|l^i\|\,.$$

Since $\in \mathbb{p}^*$, for any m, we have

$$\lim_{i,m\to\infty} \frac{a_{i+1}^m}{a_i^m} = \lim_{i,m\to\infty} \theta(\xi_n,\xi_m,\varrho)\frac{\|l^{i+1}\|}{\|l^i\|}$$

$$= \lim_{n,m\to\infty} \theta(\xi_n,\xi_m,\varrho)\rho(l) < 1.$$

Since $\lim_{n,m\to\infty} \|l\|\theta(\xi_n,\xi_m,\varrho) < 1$, the series

$$\sum_{i=1}^{m-1} l^i \prod_{j=1}^{i} \theta(\xi_j, \xi_m, \varrho)$$

converges absolutely by ratio test. Also, the series is convergent in $\mathcal{A}$, because $\mathcal{A}$ is absolutely convergent and a Banach algebra. Thus:

$$\lim_{m\to\infty} \left\| \sum_{i=1}^{m-1} l^i \prod_{j=1}^{i} \theta(\xi_j, \xi_m, \varrho) \right\| \le \lim_{m\to\infty} \sum_{i=1}^{m-1} \prod_{j=1}^{i} \theta(\xi_j, \xi_m, \varrho) \|l^i\| = \theta_{\mathcal{A}}.$$

Therefore from (4.4), We determine that, $\{\xi_n\}$ is d_{ecb_2}-Cauchy sequence in X. Due to, X is d_{ecb_2}-complete, so $\{\xi_n\}$ is d_{ecb_2}-convergent sequence in X. There exist a point $u \in X$ such that $\xi_n \to u$.

It will now be demonstrated that u serves as a fixed point of T.

Since, T is d_{ecb_2}-orbitally continuous in X. We have,

$$d_{ecb_2}(Tu, u, \varrho) \preccurlyeq \theta(Tu, u, \varrho) \begin{bmatrix} d_{ecb_2}(Tu, \xi_n, \varrho) \\ +d_{ecb_2}(Tu, \xi_n, u) \\ +d_{ecb_2}(u, \xi_n, \varrho) \end{bmatrix},$$

$$= \theta(Tu, u, \varrho) \begin{bmatrix} d_{ecb_2}(Tu, T^n\xi_0, \varrho) \\ +d_{ecb_2}(Tu, T^n\xi_0, u) \\ +d_{ecb_2}(u, \xi_n, \varrho) \end{bmatrix},$$

$$\precsim \theta(Tu, u, \varrho) \begin{bmatrix} l\, d_{ecb_2}(u, \xi_{n-1}, \varrho) \\ +l\, d_{ecb_2}(u, \xi_{n-1}, u) \\ +\ d_{ecb_2}(u, \xi_n, \varrho) \end{bmatrix},$$

$$\precsim \theta_{\mathcal{A}}, \quad \text{as } n \to \infty.$$

Thus, $d_{ecb_2}(Tu, u, \varrho) = \theta_{\mathcal{A}}$, for all $\varrho \in X$. Hence u is a fixed point of T.

Lastly, we demonstrate that the fixed point u is unique. Let v be another fixed points of T, then for all $\varrho \in X$, we have

$$d_{ecb_2}(u, v, \varrho) = d_{ecb_2}(Tu, Tv, \varrho) \precsim l\, d_{ecb_2}(u, v, \varrho),$$

That is $(e_{\mathcal{A}} - l) d_{ecb_2}(u, v, \varrho) \precsim \theta_{\mathcal{A}}$. Both sides are multiplied by

$$(e_{\mathcal{A}} - l)^{-1} = \textstyle\sum_{i=0}^{\infty} l^i \geq 0,$$

We get $d_{ecb_2}(u, v, \varrho) \precsim \theta_{\mathcal{A}}$. Thus, $d_{ecb_2}(u, v, \varrho) = \theta_{\mathcal{A}}$, this indicates that $u = v$, resulting in a conflict, thus indicating that the fixed point of T is unique.

Definition 4.7 [13] : Let $T: X \to X$ be a function. A function $G: X \to \mathbb{R}$ is considered T-orbitally lower semi-continuous at $t \in X$ if for any sequence $\{\xi_n\} \subset \mathcal{O}(\xi_0)$, where ξ_n converges to t, it holds that

$$G(t) \leq \lim_{n\to\infty} \inf G(\xi_n).$$

Theorem 4.8: Let $\mathbb{p}$ be a solid cone in $\mathcal{A}$ and $(X,\ d_{ecb_2},\ \mathcal{A})$ be a complete ECb_2MSBA. Suppose that $T: X \to X$, for each $\xi \in \mathcal{O}(\xi_0)$ where $\xi_0 \in X$, such that

$$d_{ecb_2}(T\xi, T^2\xi, \varrho) \preccurlyeq l\, d_{ecb_2}(\xi, T\xi, \varrho), \qquad (4.5)$$

where $l \in \mathbb{p}^*$, $\rho(l) < 1$ with $\lim_{n,m\to\infty} \theta(\xi_n, \xi_m, \varrho) < \frac{1}{\rho(l)}$, and $\{\xi_n\} = \{T^n\xi_0\}$, is the iterative Picard sequence produced by $\xi_0 \in X$. Then $T^n\xi_0 \to u$ ($as\ n \to \infty$). Moreover, a point u is considered fixed under the transformation T in the space X if and only if $G(v) = d_{ecb_2}(u, Tu, \varrho)$ is T -orbitally lower semi-continuous at u in X.

Proof :

Let us select an arbitrary element $\xi_0 \in X$ and define the iterative sequence $\{\xi_n\}$ by

$$\xi_0,\ \xi_1 = T\xi_0,\ \xi_2 = T\xi_1 = T^2\xi_0, \ldots,\ \xi_n = T^n\xi_0 \ \ldots$$

Now for, $\eta = T\zeta_0$, applying inequality (4.5) one after the other yields

$$
\begin{aligned}
d_{ecb_2}(T^n\xi_0, T^{n+1}\xi_0, \varrho) \\
&= d_{ecb_2}(\xi_n, \xi_{n+1}, \varrho) \\
&\preccurlyeq l^n d_{ecb_2}(\xi_0, \xi_1, \varrho) \qquad (4.6)
\end{aligned}
$$

To demonstrate that $\{\xi_n\}$ is a Cauchy sequence in X, consider the following: for every $n \in \mathbb{N}$, we have

$$
d_{ecb_2}(\xi_n, \xi_m, \varrho) \preccurlyeq \theta(\xi_n, \xi_m, \varrho) \begin{bmatrix} d_{ecb_2}(\xi_n, \xi_{n+1}, \varrho) \\ +d_{ecb_2}(\xi_n, \xi_{n+1}, \xi_m) \\ +d_{ecb_2}(\xi_{n+1}, \xi_m, \varrho) \end{bmatrix}
$$

By employing the identical method as demonstrated in the proof of Theorem 4.6, we deduce that $\{\xi_n\}$ constitutes a Cauchy sequence within X. Given that X is complete, there exists a point $u \in X$ such that $\xi_n = T^n\xi_0 \to u$. Suppose G is T-orbitally lower semi-continuous at the point u within the set X, then

for all $\varrho \in X$, we have

$$
\begin{aligned}
d_{ecb_2}(u, Tu, \varrho) &\preccurlyeq \liminf_{n\to\infty} \; d_{ecb_2}(T^n\xi_0, T^{n+1}\xi_0, \varrho) \\
&\preccurlyeq \liminf_{n\to\infty} l^n d_\theta(\xi_0, \xi_1, \varrho) \\
&= \theta_{\mathcal{A}}, \qquad (4.7)
\end{aligned}
$$

thus $Tu = u$. Therefore, u is a fixed point of T.

Conversely, let $u = Tu$ and $\xi_n \in \mathcal{O}(\xi_0)$ with $\xi_n \to u$. Then,

$$
\begin{aligned}
G(v) &= d_{ecb_2}(u, Tu, \varrho) \\
&= \theta_{\mathcal{A}} \\
&\preccurlyeq \liminf_{n\to\infty} G(\xi_n) \\
&= d_{ecb_2}(T^n\xi_0, T^{n+1}\xi_0, \varrho)\,. \qquad (4.8)
\end{aligned}
$$

Example 4.9: Consider the Banach algebra $\mathcal{A}$ and the cone $\mathbb{p}$ identical to those described in Example 3.4 and $X = [0,1]$. Define $\theta : X \times X \times X \longrightarrow [1,\infty)$ by

$$
\theta(\xi,\eta,\zeta) = \begin{cases} |\xi| + |\eta| + |\zeta| & \text{for } \xi \neq \eta \neq \zeta\,, \\ 1, & \text{otherwise.} \end{cases}
$$

Additionally, create a mapping $d_{ecb_2}: X \times X \times X \longrightarrow [0,\infty)$ as in Example 3.4. Therefore $(X,\ d_{ecb_2},\ \mathcal{A})$, is complete EC$b_2$MSBA, with $p = 2$ and the function $\theta(\xi,\eta,\zeta) < 3$. Specify $T: X \to X$ by,

$T\xi = \frac{\xi}{2}\sin\xi$, Since $u \sin u \lesssim u$ for each $t \in [0,\infty)$, for all $\xi, \eta, \varrho \in X$. Then, we have

$$
d_{ecb_2}(T\xi, T\eta, \varrho) \preccurlyeq (|\xi| + |\eta| + |\zeta|) \begin{bmatrix} \frac{\xi}{2}\sin\xi \cdot \frac{\eta}{2}\sin\eta \\ +\ \varrho \cdot \frac{\xi}{2}\sin\xi \\ +\ \varrho \cdot \frac{\eta}{2}\sin\eta \end{bmatrix}^2 e^t,
$$

$$
\preccurlyeq (|\xi| + |\eta| + |\zeta|) \begin{bmatrix} \frac{\xi\eta}{4} + \frac{\varrho\xi}{2} + \frac{\varrho\eta}{2} \end{bmatrix}^2 e^t,
$$

$$
= (|\xi| + |\eta| + |\zeta|)\frac{1}{4} \begin{bmatrix} \frac{\xi\eta}{2} + \varrho\xi + \varrho\eta \end{bmatrix}^2 e^t,
$$

$$
\preccurlyeq (|\xi| + |\eta| + |\zeta|)\frac{1}{4}\,[\xi\eta + \varrho\xi + \varrho\eta]^2 e^t,
$$

$$
= \frac{1}{4}\ d_{ecb_2}(\xi, \eta, \varrho)\,,
$$

$$
= l\, d_{ecb_2}(\xi, \eta, \varrho),
$$

That, $l = \frac{1}{4}$. Now for each $\xi \in X$, $T^n\xi = \frac{\xi}{2^n}\sin\frac{\xi}{2^n}$. Thus we obtain

$$
\begin{aligned}
&\lim_{n,m\to\infty} \theta(T^n\xi, T^m\xi, \varrho) \\
&\preccurlyeq \lim_{n,m\to\infty} \left(\left|\frac{\xi}{2^n}\sin\frac{\xi}{2^n}\right| + \left|\frac{\xi}{2^m}\sin\frac{\xi}{2^m}\right| + |\varrho| \right) \\
&\preccurlyeq \lim_{n,m\to\infty} \left(\left|\frac{\xi}{2^n}\right| + \left|\frac{\xi}{2^m}\right| + |\varrho| \right)
\end{aligned}
$$

$$\preccurlyeq 3 \prec 4 = \frac{1}{\rho(l)}.$$

Tues, $\rho(l) = \frac{1}{4} < 1$. Therefore all requirements outlined in Theorem 4.6 are met and the only fixed point of T is 0.

2. An Integral Equations Application

There exists a vast landscape of mathematical applications for fixed-point theorems, with integral equations standing out as a particularly fertile ground for their implementation. Equations of the form exemplified by (5.1) have garnered considerable attention, evidenced by the extensive research documented in works such as [3, 12, 20] and the numerous references found within. Building upon the insights presented by Kamran, et al. (2017) in their paper [28], we aim to leverage the power of Theorem 4.6 to demonstrate a key property of the Fredholm integral equation, specifically

$$\xi(t) = g(t) + \int_I M(t,s,\xi(s))ds, \quad t,s \in I, \quad I = [0,1] \quad (5.1)$$

has a unique solution in X.

Let, $X = C(I,\mathbb{R})$ constitute the space of all continuous real valued functions on $I = [0,1]$. Advised that X is $\theta_{\mathcal{A}}$-complete ECb_2MSBA, $\mathcal{A} = C(I)$, that equipped with the norm $\|\xi\| = \|\xi\|_\infty + \|\acute{\xi}\|_\infty$. Set

$\mathbb{p} = \{\xi \in \mathcal{A} : \xi(t) \geq 0, \ t \in I\}$. Take the usual multiplication, then $\mathcal{A}$ is a Banach algebra with the unit $e_{\mathcal{A}} = 1$.

Define $d_{ecb2}: X \times X \times X \to [0,\infty)$ by

$$d_{ecb_2}(\xi,\eta,\zeta) = \left[\max_{t\in I}\min\begin{Bmatrix}|\xi(t)-\eta(t)|,\\ |\xi(t)-\zeta(t)|,\\ |\eta(t)-\zeta(t)|\end{Bmatrix}\right]^2 e^t, \qquad (5.2)$$

with , $\theta: X \times X \times X \to [1,\infty)$ by

$$\theta(\xi,\eta,\zeta) = \begin{cases}|\xi| + |\eta| + |\zeta| \ \text{ for } \ \xi \neq \eta \neq \zeta, \\ 1, \qquad \text{otherwise.}\end{cases}$$

Assume that, the operator $T: X \to X$ is provided by:

$$T\xi(t) = g(t) + \int_I M(t,s,\xi(s))ds, \quad t,s \in I. \quad (5.3)$$

It is assumed that the subsequent conditions are met:

i. $g: [0,1] \to \mathbb{R}$, and $M: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions,

ii. for $t,s \in [0,1]$ and $\xi \in X$,

$$|M(t,s,\xi(s)) - M(t,s,\eta(s))| \precsim \frac{1}{2}\left\|\left[\max_{t\in I}\min\begin{Bmatrix}|\xi(s)-\eta(s)|,\\ |\xi(s)-\varrho(s)|,\\ |\eta(s)-\varrho(s)|\end{Bmatrix}\right]\right\|.$$

Theorem 5.1: Based on assumptions i and ii the Fredholm integral equation (5.1) possesses a singular solution within the set X.

Proof:

Establish the definition of the ECb_2M $d_{ecb_2}: X \times X \times X \to [0,\infty)$ as stated above by equation (5.2). Then $(X,\ d_{ecb_2},\ \mathcal{A})$ is a θ-complete ECb_2MSBA. Also we create the operator $T: X \to X$, given by equation (5.3), such that T is a contractive mapping. By using assumptions, we obtain that:

$$d_{ecb_2}(T\xi,T\eta,\varrho)(t) = \left[\max_{t\in I}\min\begin{Bmatrix}|T\xi(t)-T\eta(t)|,\\ |T\xi(t)-\varrho(t)|,\\ |T\eta(t)-\varrho(t)|\end{Bmatrix}\right]^2 e^t$$

$$\precsim [|T\xi(t) - T\eta(t)|]^2 e^t,$$

$$= \left[\left\| \begin{matrix} \int_I M(t,s,\xi(s))ds + g(t) \\ -\int_I M(t,s,\eta(s))ds - g(t) \end{matrix} \right\|\right]^2 e^t,$$

$$\precsim \left[\left\| \int_I M(t,s,\xi(s))ds - \int_I M(t,s,\eta(s))ds \right\|\right]^2 e^t,$$

$$\precsim \frac{1}{4}\left[\max \min_{t\in I} \begin{Bmatrix} |\xi(t) - T\xi(t)|, \\ |\xi(t) - \varrho(t)|, \\ |T\xi(t) - \varrho(t)| \end{Bmatrix}\right]^2 e^t,$$

$$= \frac{1}{4}\, d_{ecb2}(\xi,\eta,\varrho)(t).$$

Hence, $d_{ecb_2}(T\xi, T\eta, \varrho) \precsim l\, d_{ecb_2}(\xi,\eta,\varrho)$. Therefore, all the requirements of Theorem 5.1 are met, indicating that the mapping T possesses a fixed point. Consequently, we deduce that the Fredholm integral equation (5.1) has a singular solution in $X = C(I,\mathbb{R})$.

Conclusion

An ECb_2MSBA is a new kind of generalized Cb_2MS that we introduced in this article. In the new structure, we have also created Lipschitz maps, which provide several fixed-point theorems applicable to these spaces. The primary findings have a number of ramifications in various areas. Cb_2MSBA, for instance, implies corresponding fixed-point results when $\theta(\xi,\eta,\zeta) = s,$. Additionally, the C$b_2$M simplifies to a C2M for $\theta(\xi,\eta,\zeta) = 1$ [25]. The corresponding results at fixed points within the CEbMSBA, however, are implied by taking $\theta(\xi,\eta,\zeta) = \theta(\xi,\eta)$ [10, 11, 26]. The CbMSBA is the result of the corresponding fixed point if $\theta(\xi,\eta) = s$. For the specific case where $\theta(\xi,\eta) = 1$, the cone b-metric reduces further to a standard cone metric [9, 15, 16].

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